II. TWO-PERIOD ECONOMIES: A REVIEW

1. Introduction

These notes briefly review two-period economies. This includes the consumer's problem, the producer's problem, and general equilibrium. In what follows, I borrow freely from Chiang (1984), Farmer (1993), Feeney (1999), Obstfeld and Rogoff (1996), and Smith (1997).

2. Constrained Optimization

Throughout, we make extensive use of constrained optimization. That is, we aim to solve problems of the following form:

$$\max f(x_1, \ldots, x_n) \quad \text{subject to} \quad g(x_1, \ldots, x_n) \leq y.$$

This problem is solved using the Lagrangian:

$$L = f(x_1, \ldots, x_n) + \lambda[y - g(x_1, \ldots, x_n)],$$

where λ is a Lagrange multiplier.

Assuming an interior solution, the necessary first-order *necessary* conditions for a maximum are

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial g}{\partial x_j} = 0 \qquad \text{for} \quad j = 1, \ldots, n.$$

The second-order *sufficient* conditions for a maximum are that the bordered hessian, the matrix of second derivatives of the function $f(\cdot)$ bordered by the first derivatives of the function $g(\cdot)$, is negative definite (i.e. the bordered principal minors alternate in sign).

To simplify our life, most of our applications will deal with a simple case for which there exists an interior absolute maximum. That is, we will assume that the objective function $f(\cdot)$ is explicitely quasiconcave and that the constraint set $g(\cdot)$ is convex.

- A function $f(\cdot)$ is explicitely quasiconcave if for any pair of distinct points u and v in the domain of f, and for $0 < \theta < 1$, $f(v) > f(u) \Rightarrow f[\theta u + (1-\theta)v] > f(u)$.

† A twice continuously differentiable function $f(\cdot)$ is quasiconcave if its Hessian bordered by its first derivatives is negative definite.

- A function $g(\cdot)$ is convex only if for any pair of distinct points u and v in the domain of g, and for $0 < \theta < 1$, $\theta g(u) + (1-\theta)g(v) \geq g[\theta u + (1-\theta)v]$.

  † A twice continuously differentiable function $g(\cdot)$ is convex if its hessian is everywhere positive semidefinite.

3. Consumption

3.1. The Standard Consumer's Problem

The standard consumer's problem is

$$\max U(c_1, c_2) \quad \text{subject to} \quad p_1 c_1 + p_2 c_2 = W,$$

where c_1 an c_2 denote consumption of goods 1 and 2, p_1 and p_2 are the prices of these goods, and W is the consumer's income.

The above optimization problem is composed of an objective function $U(\cdot)$ and a constraint. The objective function is the utility function. It summarizes the consumer's preferences for goods c_1 and c_2. Your basic graduate microeconomics course should layout the conditions for this function to exist and be quasiconcave. In general, it requires that preferences adhere to the assumptions of completeness, reflexivity, transitivity, continuity, strong monotonicity, non-satiation, and convexity. The constraint is simply a convex budget set. Finally, we assume throughout that the consumer is a price taker: the consumer takes prices as given.

As stated above, the assumptions of a quasiconcave utility function and a convex budget constraint are sufficient to ensure the existence of an absolute maximum. To analyze this simple problem, then, we will only consider the first-order conditions. Proceeding with the Lagrangian, we can solve this problem as follows:

$$\max U(c_1, c_2) + \lambda(W - p_1 c_1 - p_2 c_2).$$

The first-order conditions are

$$\frac{\partial U}{\partial c_1} - \lambda p_1 = U_1(c_1, c_2) - \lambda p_1 = 0;$$

$$\frac{\partial U}{\partial c_2} - \lambda p_2 = U_2(c_1, c_2) - \lambda p_2 = 0.$$

The ratio of these conditions is

$$\frac{U_2(c_1,c_2)}{U_1(c_1,c_2)} = \frac{p_2}{p_1}.$$

The above equation has two parts. On the left is the marginal rate of substitution. It describes a consumer's willingness to substitute one good for another. On the right is the price ratio, which describes the market's willingness for the substitution. This equilibrium condition can be shown on a diagram. It is the point at which the budget set is tangent to the indifference curve. The equation of the budget line is

$$c_2 = \frac{W}{p_2} - \frac{p_1}{p_2}c_1,$$

such that its slope is $-p_1/p_2 < 0$. The indifference curve is derived from $U(c_1,c_2) =$ constant. Its slope is

$$\frac{dc_2}{dc_1} = -\frac{U_1(c_1,c_2)}{U_2(c_1,c_2)} < 0.$$

3.2. The Two-Period Consumer's Problem

The main difference between the standard problem and the two-period one is that we must allow for time.

In a dynamic framework, the consumer receives an income in periods 1 and 2. Denote these income y_1 and y_2. Then, in the first period, the consumer chooses to consume or save his income:

$$c_1 + s_1 = y_1,$$

where c_1 is the amount consumed and s_1 is the amount saved. In the second period, the consumer receives both his period income and principal plus interest on its savings. Thus, the consumer chooses to consume:

$$c_2 = y_2 + (1+r)s_1,$$

where c_2 is the amount consumed and r is the (real) rate of interest. Note that the consumer will not save in the second period, because of its finite lifetime.

The two period budgets can be consolidated to form the intertemporal budget constraint. This constraint is

$$c_1 + \frac{1}{1+r}c_2 = y_1 + \frac{1}{1+r}y_2.$$

It states that the present value (or discounted sum) of consumption equals the present value of income. There are obvious similarities between the intertemporal budget constraint and the budget constraint of the standard consumer's problem. We can write this last constraint as

$$c_1 + \frac{p_2}{p_1} c_2 = \frac{W}{p_1}.$$

A comparison suggests that $1/(1+r) = p_2/p_1$. That is, *the market discount factor is the relative price of second-period consumption.*

Now that we have dealt with the budget, we wish to give a dynamic interpretation to the utility function. As stated before, goods c_1 and c_2 now simply correspond to consumption in periods 1 and 2. In most application, we will assume that the utility function is time additive:

$$U(c_1, c_2) = u(c_1) + \beta u(c_2),$$

where $u(c)$ is the period (or instantaneous) utility function with $u'(c) > 0$ and $u''(c) \leq 0$, $0 < \beta = 1/(1+\rho) < 1$ is the subjective discount factor, and ρ is the rate of time preference. Large values of ρ show a preference for consumption today, as the implied low value for β indicate a large discount on future utility.

The two-period consumer's problem is

$$\max u(c_1) + \beta u(c_2) \quad \text{subject to} \quad c_1 + \frac{1}{1+r} c_2 = y_1 + \frac{1}{1+r} y_2.$$

The Lagrangian is

$$L = u(c_1) + \beta u(c_2) + \lambda \left[y_1 + \frac{1}{1+r} y_2 - c_1 - \frac{1}{1+r} c_2 \right].$$

The first-order conditions can be summarized by

$$\beta \frac{u'(c_2)}{u'(c_1)} = \frac{1}{1+r},$$

where $u'(c) = \partial u(c)/\partial c$. This condition (the so-called *Euler* equation) simply states that the intertemporal marginal rate of substitution equals the market discount factor. As before, this can simply be represented by a diagram. The budget line is

$$c_2 = (1+r)y_1 + y_2 - (1+r)c_1,$$

such that its slope is $-(1+r) < 0$. The indifference curve is derived from $u(c_1) + \beta u(c_2) =$ constant. Its slope is

$$\frac{dc_2}{dc_1} = -\frac{u'(c_1)}{\beta u'(c_2)} < 0.$$

This slope is an increasing function of first period consumption (i.e. as c_1 increases the slope becomes flatter):

$$\begin{aligned}\frac{d^2c_2}{dc_1^2} &= -\frac{u''(c_1)}{\beta u'(c_2)} + \beta u''(c_2)\frac{u'(c_1)}{[\beta u'(c_2)]^2}\frac{dc_2}{dc_1} \\ &= -\frac{u''(c_1)}{\beta u'(c_2)} - \beta u''(c_2)\frac{[u'(c_1)]^2}{[\beta u'(c_2)]^3} \geq 0.\end{aligned}$$

Two interesting cases come out of our analysis. The first, *consumption smoothing*, occurs when $\rho = r$, such that $\beta = 1/(1+r)$. It implies that $u'(c_1) = (1+r)\beta u'(c_2) = u'(c_2)$ and that $c_1 = c_2$. The second, *consumption tilting*, occurs when $\rho \neq r$. For example, assume that $\rho < r$ such that $\beta > 1/(1+r)$: the consumer discounts the future less than the market or the consumer is less impatient than the market. In that case, $u'(c_1) = (1+r)\beta u'(c_2) > u'(c_2)$ and $c_1 < c_2$ (because marginal utility is decreasing).

Finally, the consumer's problem can be used to yield demand functions. In general, demand functions are found by *i.* solving the consumer's problem and *ii.* substituting the first-order conditions in the budget constraint. We show how this can be accomplished using a simple example:

$$\max u(c_1) + \beta u(c_2) \quad \text{subject to} \quad c_1 + \frac{1}{1+r}c_2 = y_1 + \frac{1}{1+r}y_2,$$

where

$$u(c) = \frac{c^{1-1/\sigma} - 1}{1 - 1/\sigma}.$$

The first-order conditions imply

$$\beta\left(\frac{c_1}{c_2}\right)^{1/\sigma} = \frac{1}{1+r}.$$

This condition can be substituted in the budget to yield the demand functions:

$$c_1 = \left[\frac{1}{1+(1+r)^{\sigma-1}\beta^{\sigma}}\right]\left(y_1 + \frac{1}{1+r}y_2\right);$$

$$c_2 = \left[\frac{(1+r)^{\sigma}\beta^{\sigma}}{1+(1+r)^{\sigma-1}\beta^{\sigma}}\right]\left(y_1 + \frac{1}{1+r}y_2\right).$$

The above framework can be used to discuss the relation between interest rate and the price of a bond. Assume that the consumer has two vehicles for his savings. First, he can purchase an amount b_1 of a bond that has the same price as first period consumption and remits $(1+r)$ unit of second period consumption. Second, he can purchase an amount

a of a bond at price q that remits a fixed d units of second period consumption. In that case, is period 1 budget is

$$c_1 + b_1 + qa_1 = y_1.$$

The period 2 budget is

$$c_2 = y_2 + (1 + r)b_1 + da_1.$$

We can write the optimization problem for this case simply as

$$\max u(y_1 - b_1 - qa_1) + \beta u\left(y_2 + (1 + r)b_1 + da_1\right).$$

The first-order conditions in that case are summarized by

$$\beta \frac{u'(c_2)}{u'(c_1)} = \frac{1}{1 + r} = \frac{q}{d}.$$

The above implies an inverse relation between bonds prices and interest rates:

$$1 + r = \frac{d}{q}.$$

Also, it is clear that $s_1 = b_1 + qa_1$.

3.3. Savings and the Elasticity of Intertemporal Substitution

So far, we have discussed consumption. It is however interesting to figure out how savings is related to the interest rate. In general, we think that an increase in the interest rate increases the return to savings, and thus leads to more savings [draw the savings schedule]. In what follows, we verify this conjecture. To do so, however, we must define the elasticity of intertemporal substitution.

We define the elasticity of intertemporal substitution as

$$\sigma(c) = -\frac{u'(c)}{cu''(c)}.$$

This elasticity describes the curvature of the utility function.

Macroeconomists use a variety of utility functions. The most popular is the so-called constant relative risk aversion (CRRA) utility. In that case, the period utility is

$$u(c) = \begin{cases} \frac{c^{1-1/\sigma}-1}{1-1/\sigma}; & \\ \ln(c) & \text{if } \sigma = 1. \end{cases}$$

For this utility function, the elasticity of intertemporal substitution is constant:

$$\sigma(c) = \sigma.$$

This elasticity of substitution can inform us of the response of relative consumption to an interest rate change. Using the Euler equation for our CRRA case, we can show that

$$\frac{c_2}{c_1} = \beta^{\sigma}(1+r)^{\sigma}.$$

Taking logs and a total differential yields:

$$d\log\left(\frac{c_2}{c_1}\right) = \sigma d\log(1+r).$$

Thus, for large values of σ, the response would be large.

We can now verify our initial intuition. Using the budget to eliminate both c_1 and c_2 in the Euler equation, we write

$$\beta^{\sigma}(1+r)^{\sigma}(y_1 - s_1) = y_2 + (1+r)s_1$$

Then,

$$\frac{ds_1}{dr} = \frac{\left[\sigma\beta^{\sigma}(1+r)^{(\sigma-1)}(y_1 - s_1) - s_1\right]}{\left[(1+r) + \beta^{\sigma}(1+r)^{\sigma}\right]} = \frac{\left[\sigma c_2/(1+r) - s_1\right]}{\left[(1+r) + c_2/c_1\right]}.$$

Clearly, a rise in the interest rate has an ambiguous effect on savings.

The previous result should not shock anyone. There are several channels through which an increase in the rate of interest affects savings. These are the substitution effect, the income effect, and the wealth effect.

i. The substitution effect implies that a rise in the interest rate makes savings more attractive. That is, an increase in r is a reduction in the price of second period consumption ($p_2/p_1 = 1/(1+r)$). This should stimulate second period consumption. For a given present value of income, this can only be achieved by a rise in savings.

ii. The income effect implies that a rise in the interest rate increases future consumption for a given present value of income and savings. In general, the consumer would spread this increase to consumption in both periods. The rise in first-period consumption reduces savings ($s_1 = y_1 - c_1$).

iii. The wealth effect implies that a rise in the interest rate increases the market discount factor that reduces the present value of income ($y_1 + \frac{1}{1+r}y_2$). This lowers first-period consumption and increases savings.

All three effects are summarized in the demand function

$$c_1 = \left[\frac{1}{1+(1+r)^{\sigma-1}\beta^{\sigma}}\right]\left(y_1 + \frac{1}{1+r}y_2\right).$$

The substitution and income effects are related to the first term on the right side and the wealth effect to the second term. Clearly, there are cases where the substitution effect dominates the income effect ($\sigma > 1$), and cases where it does not ($\sigma \leq 1$).

4. A Pure Exchange Economy

4.1. The Equilibrium

A pure exchange economy is one in which there is no production, and consumers only trade their endowments. In what follows, we assume that aggregate demand, the sum of all the consumers' individual demand functions, can be represented by the demand function of only one consumer. We call this consumer the representative agent. Finally, we will proceed informally, and let your graduate microeconomics course deal with questions of existence and uniqueness of this equilibrium.

The essential elements of our economic environment are

1. Preferences: $U(c_1, c_2) = u(c_1) + \beta u(c_2)$
2. Endowments: y_1 and y_2
3. Market Clearing Conditions: $c_1 = y_1$ and $c_2 = y_2$

The first element is simply the preferences of our representative agent. The second element, endowments, represents the income of each individual. Here, we assume that the representative agent has a tree in its backyard that yields fruits. He gets an amount of y_1 fruits in the first period and y_2 fruits in the second period. The final element shows the market clearing conditions. This condition simply imposes that the demand for fruits, c_i, equals the supply of fruits, y_i.

Our goal is to find the equilibrium for this economy. The pure exchange equilibrium is an allocation $\{c_1, c_2\}$ and prices $\{p_1, p_2\}$ such that *i.* the consumer maximizes its utility subject to his budget constraint and that *ii.* markets clear. Note that I have used prices p_1 and p_2 instead of r. You will recall from your intermediate microeconomic theory courses, however, that we can only identify 1 of those two prices (or a ratio of them). That is, one of the two goods must be the numeraire. In what follows, we use first-period consumption to be the numeraire.

The above definition itself suggests how to proceed to solve for the equilibrium. The first step is to solve the consumer's problem and find the demand functions. The second step is to impose the market clearing conditions. This approach should allow us to solve for equilibrium values of c_1, c_2, and r. We show this approach using our example with CRRA preferences. The consumer's problem is

$$\max u(c_1) + \beta u(c_2) \quad \text{subject to} \quad c_1 + \frac{1}{1+r} c_2 = y_1 + \frac{1}{1+r} y_2.$$

The first-order conditions are summarized by

$$\beta \left(\frac{c_1}{c_2} \right)^{1/\sigma} = \frac{1}{1+r}.$$

The demand functions are:

$$c_1 = \left[\frac{1}{1+(1+r)^{\sigma-1}\beta^{\sigma}}\right]\left(y_1 + \frac{1}{1+r}y_2\right);$$

$$c_2 = \left[\frac{(1+r)^{\sigma}\beta^{\sigma}}{1+(1+r)^{\sigma-1}\beta^{\sigma}}\right]\left(y_1 + \frac{1}{1+r}y_2\right).$$

The market clearing conditions are:

$$c_1 = y_1;$$

$$c_2 = y_2.$$

At this point, we have 2 demand functions and 2 market clearing conditions, but only three unknowns c_1, c_2, and r. This is simply an illustration of Walras's law: one of these equations is redundant. Substituting $c_1 = y_1$ in the first demand function yields

$$\frac{1}{1+r} = \beta\left(\frac{y_1}{y_2}\right)^{1/\sigma}.$$

Then, our pure exchange equilibrium is defined by an allocation

$$c_1 = y_1;$$

$$c_2 = y_2;$$

and a price ratio

$$\frac{p_2}{p_1} = \frac{1}{1+r} = \beta\left(\frac{y_1}{y_2}\right)^{1/\sigma}.$$

The above example suggests a simple shortcut. That is, in the representative agent pure exchange economy, there is no need to find the demand functions. The equilibrium is found as follows.

$$\max u(c_1) + \beta u(c_2) \quad \text{subject to} \quad c_1 + \frac{1}{1+r}c_2 = y_1 + \frac{1}{1+r}y_2.$$

The first-order conditions can be summarized by

$$\beta\frac{u'(c_2)}{u'(c_1)} = \frac{1}{1+r}.$$

Imposing market clearing, the equilibrium is

$$c_1 = y_1;$$

$$c_2 = y_2;$$
$$\frac{1}{1+r} = \beta \frac{u'(y_2)}{u'(y_1)}.$$

This equilibrium can be shown using a diagram. Note that the budget line must go through the pair $\{y_1, y_2\}$. Also note that, in equilibrium, nobody saves.

4.2. Macroeconomics

The main question at this point is 'How does this relate to macroeconomics?' The simple answer is that we have derived aggregate consumption and the real interest rate for an economy with no production, no government, and no trading partner. To see this, recall the national account identity: $Y = C + I + G + X - M$, where Y denotes gross national product (GNP), C is aggregate consumption, I is aggregate investment, G is government expenditures, X is exports, and M imports. If we abstract from investment, government, and trading partners, the identity reduces to $Y = C$. Thus, in our simple pure exchange economy, GNP is y_1 in the first period and y_2 in the second. Aggregate consumption is c_1 in period 1 and c_2 in period 2. Also note that in general, we would have that $S = Y - C - T = I + G - T + X - M$. In our case, there is no investment, no government, and no foreign trade, which explains why there is no savings in equilibrium.

5. A Simple Production Economy

5.1. The Economic Environment

We wish to extend our macroeconomic analysis to the case of a production economy. As a first step, we will consider the labor-leisure choice only. In subsequent sections, we will consider capital and investment. We retain our assumption of a representative consumer, and add a representative firm to the economy. This firm will produce consumption goods using labor only. In this section, we abstract from population growth.

Our economic environment is described by:

1. Preferences: $u(c_1, \ell_1) + \beta u(c_2, \ell_2)$

   We assume that the period utility function has the following characteristics: $u_1(c, \ell) > 0$, $u_2(c, \ell) > 0$, $u_{11}(c, \ell) < 0$, $u_{22}(c, \ell) < 0$, and $u_{12}(c, \ell) = 0$ (simplifying assumption).
2. Technology: $y_1 = f(n_1)$ and $y_2 = f(n_2)$

   We assume that the production function exhibits $f'(n) > 0$ and $f''(n) < 0$ (decreasing returns).

3. Goods Market Clearing Conditions: $c_1 = y_1$ and $c_2 = y_2$
4. Factors Market Clearing Conditions: $\ell_1 + n_1 = 1$ and $\ell_2 + n_2 = 1$

Our main addition is the production function: $y = f(n)$ (this function can be shown using a diagram). In the context of our environment, ℓ is leisure, n is hours worked or labor supplied. Also, we have fixed the endowment of time per period to unity. We wish to impose a number of restrictions on this production function. These are:

1. No labor, no output: $f(0) = 0$.
2. It is increasing and concave: $f'(n) > 0$ and $f''(n) < 0$.
3. Inada conditions apply: $\lim_{n\to 0} f'(n) = \infty$ and $\lim_{n\to\infty} f'(n) = 0$.

A good example of a function that satisfies these conditions is $f(n) = An^{\alpha}$ for $0 < \alpha < 1$. This function can be shown on a diagram.

There are basically two ways to find the equilibrium of this production economy. The first is via a competitive equilibrium. The second is via a planner's problem.

5.2. The Competitive Equilibrium

The idea behind the competitive equilibrium is to let the price system allocate resources. As before, we define the equilibrium as an allocation $\{c_1, c_2, y_1, y_2, n_1, n_2, \ell_1, \ell_2\}$ and prices $\{r, w_1, w_2\}$ such that the consumer maximizes its utility subject to budget, the firm maximizes profits, and markets clear. We again use first-period consumption as our numeraire and define w_1 and w_2 as real wages.

The consumer's problem is:

$$\max u(c_1, 1 - n_1) + \beta u(c_2, 1 - n_2)$$

subject to

$$c_1 + s_1 = w_1 n_1 + \pi_1$$

$$c_2 = w_2 n_2 + (1 + r)s_1 + \pi_2$$

The two period budget constraint can be consolidated to yield an intertemporal budget:

$$c_1 + \frac{1}{1+r}c_2 = w_1 n_1 + \pi_1 + \frac{1}{1+r}\left(w_2 n_2 + \pi_2\right).$$

The Lagrangian is

$$L = u(c_1, 1 - n_1) + \beta u(c_2, 1 - n_2) + \lambda\left(w_1 n_1 + \pi_1 + \frac{1}{1+r}(w_2 n_2 + \pi_2) - c_1 - \frac{1}{1+r}c_2\right).$$

These first-order conditions are summarized by

$$\frac{u_2(c_1, 1-n_1)}{u_1(c_1, 1-n_1)} = w_1;$$

$$\frac{u_2(c_2, 1-n_2)}{u_1(c_2, 1-n_2)} = w_2;$$

$$\beta \frac{u_1(c_2, 1-n_2)}{u_1(c_1, 1-n_1)} = \frac{1}{1+r}.$$

The first two of these conditions state that the marginal rate of substitution between leisure and consumption equals the wage rate. These can be shown on a diagram. These are static or intratemporal conditions. The last condition states that the intertemporal marginal rate of substitution equals the market discount factor.

The firm's problem is to maximize the present value of its profits:

$$\max \pi_1 + \frac{1}{1+r}\pi_2,$$

where

$$\pi_1 = f(n_1) - w_1 n_1;$$

$$\pi_2 = f(n_2) - w_2 n_2;$$

As for the consumer, the firm is a price taker. That is, the firm takes interest rates and wages rates as given. The firm's problem is

$$\max f(n_1) - w_1 n_1 + \frac{1}{1+r}\left(f(n_2) - w_2 n_2\right).$$

The first-order conditions are

$$f'(n_1) = w_1;$$

$$f'(n_2) = w_2.$$

These intratemporal conditions simply state that the marginal product of labor equals the wage rate.

To solve for our equilibrium, we require values for $\{c_1, c_2, y_1, y_2, n_1, n_2, \ell_1, \ell_2\}$ and $\{r, w_1, w_2\}$. There are thus 11 unknowns that must be solved using the following 11 equations:

$$\frac{u_2(c_1, 1-n_1)}{u_1(c_1, 1-n_1)} = w_1 = f'(n_1);$$

$$\frac{u_2(c_2, 1-n_2)}{u_1(c_2, 1-n_2)} = w_2 = f'(n_2);$$

$$\beta \frac{u_1(c_2, 1-n_2)}{u_1(c_1, 1-n_1)} = \frac{1}{1+r}.$$

$$c_1 = y_1 = f(n_1) \qquad \text{and} \qquad c_2 = y_2 = f(n_2);$$

$$\ell_1 + n_1 = 1 \qquad \text{and} \qquad \ell_2 + n_2 = 1.$$

5.3. The Planner's Problem

The planner's problem is to find the optimal allocation of resources. Its problem is

$$\max u(c_1, 1-n_1) + \beta u(c_2, 1-n_2)$$

subject to

$$c_1 = f(n_1)$$

$$c_2 = f(n_2)$$

The Lagrangian is

$$L = u(c_1, 1-n_1) + \beta u(c_2, 1-n_2) + \lambda_1 \left(f(n_1) - c_1\right) + \lambda_2 \left(f(n_2) - c_2\right)$$

These first-order conditions can be rearranged to yield:

$$\frac{u_2(c_1, 1-n_1)}{u_1(c_1, 1-n_1)} = f'(n_1);$$

$$\frac{u_2(c_2, 1-n_2)}{u_1(c_2, 1-n_2)} = f'(n_2);$$

These conditions state that the marginal rate of substitution between leisure and consumption equals the marginal product of labor.

Thus, the planner must find values for 8 unknowns, $\{c_1, c_2, y_1, y_2, n_1, n_2, \ell_1, \ell_2\}$, using the 8 equations:

$$\frac{u_2(c_1, 1-n_1)}{u_1(c_1, 1-n_1)} = f'(n_1);$$

$$\frac{u_2(c_2, 1-n_2)}{u_1(c_2, 1-n_2)} = f'(n_2);$$

$$c_1 = y_1 = f(n_1) \qquad \text{and} \qquad c_2 = y_2 = f(n_2);$$

$$\ell_1 + n_1 = 1 \qquad \text{and} \qquad \ell_2 + n_2 = 1.$$

5.4. *Discussion*

A comparison of the equations required to solve for the equilibrium in both the competitive equilibrium and the planner's problem demonstrate that the allocation is the same under both solution method. Given that the planner's problem yields the Pareto optimal allocation, it follows that the competitive equilibrium is also Pareto optimal.

This fact is just an example of the fundamental theorems of welfare economics. There are two such theorems. I will leave the proofs of these for your graduate microeconomic instructor. The first theorem is:

First Welfare Theorem Every competitive equilibrium is Pareto optimal.

It states that, under certain conditions, the allocation under the competitive equilibrium is Pareto optimal. That is, it is not possible to find another allocation that would make one person better off, while making nobody worse off. The conditions ensure that there are no imperfections, such as externalities, public goods, and altruism.

The second theorem is:

Second Welfare Theorem Every Pareto optimum can be decentralized as a competitive equilibrium.

This one states that the planner's allocation can be supported as a competitive equilibrium. This last theorem might require the existence of a tax and transfer system to ensure that the distribution of endowments is compatible with the desired allocation of resources.

6. A Production Economy with Capital Accumulation

6.1. *Production and Investment*

The economy that we are about to discuss has a more complex production technology. The firm produces output using both labor and capital as inputs.

The first addition to our economy will be the production function. In general, the production function is of the form:

$$Y = F(K, N),$$

where Y is output, K is the capital stock, and N is hours worked. We assume that this function has constant returns to scale. That is, it is linear homogenous. A function is linear homogenous or homogenous of degree 1 when

$$JF(K, N) = F(JK, JN).$$

Linear homogenous functions have four useful properties:

Property I If the function $Y = F(K,N)$ is linearly homogenous, then it can be written in terms of the capital-labor ratio: $k = K/N$.

This is easy to show:

$$Y = NF\left(\frac{K}{N}, 1\right) = Nf(k)$$

or $y = f(k)$ where $y = Y/N$.

Property II If the function $Y = F(K,N)$ is linearly homogenous, then the average products of labor and capital can be written in terms of the capital-labor ratio k.

The average products are

$$AP_N = \frac{Y}{N} = f(k)$$

$$AP_K = \frac{Y}{K} = \frac{Y}{N}\frac{N}{K} = \frac{f(k)}{k}$$

Property III If the function $Y = F(K,N)$ is linearly homogenous, then the marginal products of labor and capital can be written in terms of the capital-labor ratio k.

To show this, first note that

$$\frac{\partial k}{\partial N} = \frac{\partial K/N}{\partial N} = -\frac{K}{N^2} \qquad \text{and} \qquad \frac{\partial k}{\partial K} = \frac{\partial K/N}{\partial K} = \frac{1}{N}.$$

Then, the marginal products are

$$\begin{aligned} MP_N = F_2(K,N) &= \frac{\partial Nf(k)}{\partial N} \\ &= f(k) + Nf'(k)\frac{\partial k}{\partial N} = f(k) - kf'(k); \end{aligned}$$

$$MP_K = F_1(K,N) = \frac{\partial Nf(k)}{\partial K} = f'(k).$$

Property IV Euler's Theorem If the function $Y = F(K,N)$ is linearly homogenous, then

$$Y = KF_1(K,N) + NF_2(K,N).$$

This is shown as follows:

$$\begin{aligned} Y &= KF_1(K,N) + NF_2(K,N) \\ &= Kf'(k) + N\left(f(k) - kf'(k)\right) \\ &= Kf'(k) + Nf(k) - Kf'(k) \\ &= Nf(k). \end{aligned}$$

In what follows, we impose a number of restrictions on the function $f(k)$. These are:

1. No capital, no output: $f(0) = 0$.
2. It is increasing and concave: $f'(k) > 0$ and $f''(k) < 0$.
3. Inada conditions apply: $\lim_{k \to 0} f'(k) = \infty$ and $\lim_{k \to \infty} f'(k) = 0$.

An example of a production function that satisfies these properties is the Cobb-Douglas function ($0 < \alpha < 1$):

$$Y = F(K, N) = AK^{\alpha}N^{1-\alpha}.$$

In its intensive form it is

$$y = f(k) = Ak^{\alpha}.$$

It can be shown on a diagram.

The second addition is capital accumulation. For this addition, the effect of time is important. We will assume that capital accumulation involves a time-to-build component. That is, investment I_1 today only increases the capital stock tomorrow:

$$K_2 = I_1 + (1 - \delta)K_1,$$

where δ is the rate of depreciation. This accumulation equation can also be transformed in per capita as follows:

$$\frac{N_2}{N_1}\frac{K_2}{N_2} = \frac{I_1}{N_1} + (1 - \delta)\frac{K_1}{N_1}$$

or

$$\gamma_n k_2 = x_1 + (1 - \delta)k_1.$$

In what follows, we abstract from population growth and assume that labor is supplied inelastically. Accordingly, we set $N_1 = N_2 = 1$ and $\gamma_n = 1$. Our accumulation equation then becomes

$$k_2 = x_1 + (1 - \delta)k_1.$$

In this case, our economic environment is described by:

1. Preferences: $u(c_1) + \beta u(c_2)$
2. Technology: $Y_1 = F(K_1, N_1)$ and $Y_2 = F(K_2, N_2)$
3. Capital Accumulation: $K_2 = I_1 + (1 - \delta)K_1$, $0 = I_2 + (1 - \delta)K_2$, and K_1 given.
3. Goods Market Clearing Conditions: $c_1 + I_1 = Y_1$ and $c_2 + I_2 = Y_2$
4. Factor Market Clearing Conditions: $n_1 = 1$ and $n_2 = 1$.

Note that fixing labor supply to unity implies that per capita variables are the variables themselves. As before, the equilibrium of this production economy can be found either via a competitive equilibrium or via a planner's problem.

6.2. The Competitive Equilibrium

The equilibrium is an allocation $\{c_1, c_2, n_1, n_2, x_1, y_1, y_2, k_2\}$ and prices $\{r, w_1, w_2, q\}$ such that the consumer maximizes its utility subject to budget, the firm maximizes profits, and markets clear. We again use first-period consumption as our numeraire.

The consumer's problem is:

$$\max u(c_1) + \beta u(c_2)$$

subject to

$$c_1 + b_1 + qa_1 = w_1 n_1 + (d_1 + q)a_0;$$

$$c_2 = w_2 n_2 + (1 + r)b_1 + d_2 a_1,$$

where a_0 is given. It is worthwhile to discuss the two budget constraint. First, the consumer supplies labor inelastically, such that $n_1 = n_2 = 1$. Second, there are two different assets that the consumer purchases. In the first period, the consumer can purchase a bond that pays $(1 + r)$ units of second period consumption and shares of the representative firm at price q_1 that remits dividends in the form of d_2 units of second period goods. The first period income stems from his labor income and his previous holdings of shares. An alternative specification of the first-period budget might remove confusions:

$$c_1 + b_1 + q(a_1 - a_0) = w_1 n_1 + d_1 a_0.$$

This states that the consumer changes his holdings of shares and receives income from his current shares. This market for shares will add a market clearing condition later on.

The optimization problem is

$$\max u\left(w_1 + (d_1 + q)a_0 - b_1 - qa_1\right) + \beta u\left(w_2 + (1 + r)b_1 + d_2 a_1\right).$$

The first-order conditions are summarized by

$$\beta \frac{u'(c_2)}{u'(c_1)} = \frac{1}{1 + r} = \frac{q}{d_2}.$$

The firm's problem is to maximize its value, which is the sum of its current dividends and share values:

$$V = (d_1 + q)A_0,$$

where A_0 is the number of shares initially issued. We assume that the firm issues a constant number of shares that we normalize to unity: $A_0 = A_1 = 1$. Clearly, the consumer's first-order conditions imply that $q = d_2/(1 + r)$. Then, the firm's problem is

$$\max V = \max d_1 + \frac{1}{1 + r} d_2,$$

where

$$d_1 = F(K_1, n_1) - I_1 - w_1 n_1 = F(K_1, n_1) - K_2 + (1-\delta)K_1 - w_1 n_1;$$

$$d_2 = F(K_2, n_2) - I_2 - w_2 n_2 = F(K_2, n_2) + (1-\delta)K_2 - w_2 n_2.$$

The optimization is

$$\max F(K_1, n_1) - K_2 + (1-\delta)K_1 - w_1 n_1 + \frac{1}{1+r}\left[F(K_2, n_2) + (1-\delta)K_2 - w_2 n_2\right].$$

The first-order conditions are

$$(1+r) = F_1(K_2, n_2) + (1-\delta) = f'(k_2) + (1-\delta);$$

$$w_1 = F_2(K_1, n_1) = f(k_1) - k_1 f'(k_1);$$

$$w_2 = F_2(K_2, n_2) = f(k_2) - k_2 f'(k_2).$$

The first condition states that the firm will invest to the point where the marginal product of capital equals the gross rate of interest. This condition can be used to argue that an increase in the rate of interest reduces investment:

$$\frac{dk_2}{dr} = \frac{1}{f''(k_2)} < 0,$$

where $k_2 = K_2/n_2 = K_2$ ($n_2 = 1$). The last two conditions state that the wage rate equals the marginal product of labor.

To solve for our equilibrium, we require values for $\{c_1, c_2, n_1, n_2, a_1, x_1, x_2, y_1, y_2, k_2\}$ and $\{r, w_1, w_2, q\}$. There are thus 14 unknowns that must be solved using the following 14 equations:

$$\beta\frac{u'(c_2)}{u'(c_1)} = \frac{1}{1+r} = \frac{q}{d_2};$$

$$f'(k_2) + (1-\delta) = 1 + r;$$

$$w_1 = f(k_1) - k_1 f'(k_1);$$

$$w_2 = f(k_2) - k_2 f'(k_2);$$

$$c_1 + x_1 = y_1 = f(k_1) \qquad \text{and} \qquad c_2 + x_2 = y_2 = f(k_2);$$

$$k_2 = x_1 + (1-\delta)k_1 \qquad \text{and} \qquad 0 = x_2 + (1-\delta)k_1;$$

$$n_1 = n_2 = a_1 = 1.$$

6.3. The Planner's Problem

As before, the planner's problem is to find the optimal allocation of resources. Its problem is

$$\max u(c_1) + \beta u(c_2)$$

subject to

$$c_1 + k_2 = f(k_1) + (1-\delta)k_1;$$

$$c_2 = f(k_2) + (1-\delta)k_2.$$

The optimization problem is

$$\max u\left(f(k_1) + (1-\delta)k_1 - k_2\right) + \beta\left(f(k_2) + (1-\delta)k_2\right).$$

The first-order condition is

$$\beta\frac{u'(c_2)}{u'(c_1)} = \frac{1}{f'(k_2) + (1-\delta)}$$

To solve for our equilibrium, the planner solves for $\{c_1, c_2, n_1, n_2, x_1, x_2, y_1, y_2, k_2\}$. There are thus 9 unknowns that must be solved using the following 9 equations:

$$\beta\frac{u'(c_2)}{u'(c_1)} = \frac{1}{f'(k_2) + (1-\delta)}$$

$$c_1 + x_1 = y_1 = f(k_1) \qquad \text{and} \qquad c_2 + x_2 = y_2 = f(k_2);$$

$$k_2 = x_1 + (1-\delta)k_1 \qquad \text{and} \qquad 0 = x_2 + (1-\delta)k_1;$$

$$n_1 = n_2 = 1.$$

6.4. Discussion

Once again, a comparison of the equations required to solve for the equilibrium in both the competitive equilibrium and the planner's problem demonstrate that the allocation is Pareto optimal.

The allocation can be displayed on a diagram. The first element is the production possibility frontier. This frontier is

$$c_2 = f\left(f(k_1) + (1-\delta)k_1 - c_1\right) + (1-\delta)\left(f(k_1) + (1-\delta)k_1 - c_1\right).$$

Its slope is

$$\frac{dc_2}{dc_1} = -\left[f'(k_2) + (1-\delta)\right] < 0$$

This slope is a decreasing function of first period consumption (i.e. as c_1 increases the slope becomes steeper):

$$\frac{d^2 c_2}{dc_1^2} = f''(k_2) < 0.$$

The second element is the indifference curve. It is derived from $u(c_1) + \beta u(c_2) =$ constant. Its slope is

$$\frac{dc_2}{dc_1} = -\frac{u'(c_1)}{\beta u'(c_2)} < 0.$$

This slope is an increasing function of first period consumption (i.e. as c_1 increases the slope becomes flatter):

$$\begin{aligned}\frac{d^2 c_2}{dc_1^2} &= -\frac{u''(c_1)}{\beta u'(c_2)} + \beta u''(c_2)\frac{u'(c_1)}{[\beta u'(c_2)]^2}\frac{dc_2}{dc_1} \\ &= -\frac{u''(c_1)}{\beta u'(c_2)} - \beta u''(c_2)\frac{[u'(c_1)]^2}{[\beta u'(c_2)]^3} \geq 0.\end{aligned}$$

We can relate this case to our national account identity. In this last economy, we have consumption and investment: $Y = C + I$. It follows that savings must equal investment $S = Y - C = I$. In our economy, aggregate savings in the first period is $Y_1 - c_1 = I_1$.

We can also use our competitive equilibrium to figure out the price of a firm's share and the firm's value. The price of a firm's share is

$$\begin{aligned}q &= \frac{d_2}{1+r} = \frac{1}{1+r}\left[F(K_2, n_2) + (1-\delta)K_2 - w_2 n_2\right] \\ &= \frac{1}{1+r}\left[K_2 F_1(K_2, n_2) + n_2 F_2(K_2, n_2) + (1-\delta)K_2 - w_2 n_2\right] \\ &= \frac{1}{1+r}\left[F_1(K_2, n_2) + (1-\delta)\right] K_2 \\ &= \frac{1}{1+r}\left[f'(k_2) + (1-\delta)\right] K_2 \\ &= K_2.\end{aligned}$$

Thus, the price of the share is the value of the capital stock. Therefore, purchasing the share of a firm is like purchasing a claim on its capital stock. The firm's value is

$$V = q + d_1 = Y_1 + (1-\delta)K_1 - w_1 n_1.$$

Finally, using the consumer's first period budget

$$c_1 + b_1 + q a_1 = w_1 n_1 + (d_1 + q) a_0,$$

we find

$$c_1 + b_1 + K_2 = Y_1 + (1-\delta)K_1$$

given that $a_1 = a_0 = 1$. Clearly, the national income identity $(c_1 + I_1 = Y_1)$ suggests that $b_1 = 0$. In this economy, there is only one vehicle for aggregate savings and it is via the firm's shares and thus its capital stock.

7. What Have We Learned About Macroeconomics?

1. The aggregate consumption function depends on the discounted sum of future income:

$$c_1 = \left[\frac{1}{1+(1+r)^{\sigma-1}\beta^{\sigma}}\right]\left(y_1 + \frac{1}{1+r}y_2\right).$$

2. The market discount factor, and thus the real interest rate, is related to the relative price of future consumption:

$$\frac{1}{1+r} = \frac{p_2}{p_1}.$$

   Both preferences and technology affect it:

$$\beta\frac{u'(c_2)}{u'(c_1)} = \frac{1}{1+r} = \frac{1}{f'(k_2)+(1-\delta)}.$$

3. There is an inverse relation between bonds prices and interest rates.

$$1 + r = \frac{d}{q}.$$

4. An increase in the wage rate increases labor supply and reduces labor demand. Thus, aggregate employment and the real wages respond to both preferences and technology:

$$\frac{u_2(c_1, 1-n_1)}{u_1(c_1, 1-n_1)} = w_1 = f'(n_1);$$

5. Investment responds negatively to increases in the interest rate and positively to increases in the marginal product of capital (partial equilibrium).
6. In an economy with no government and no trading partners, saving equals investment: $s_1 = I_1$. Moreover, it must be the case that the only vehicle for aggregate savings is the capital stock. That is, purchasing the share of a firm is like purchasing a claim on its capital stock.

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