Introduction to Dynamic Processes

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Summary

1. Major characteristics of a deterministic dynamic process
2. Major characteristics of a stochastic dynamic process
3. Some examples of different types of dynamics
4. 2-Dimensional dynamic processes
5. Nonlinear dynamic processes
6. Linearization (linear approximations) of nonlinear processes
7. What have we learned?

I – Major characteristics of a deterministic dynamic process

Dynamics and how to model it

- In modern macroeconomics, everything is discussed using **dynamics**
- **Examples:**

$$\begin{aligned} k_t &= 10 + 0.5 \cdot k_{t-1} && \text{(Difference Eq.)} \\ & \quad \mathit{or} \\ \underbrace{k_t - k_{t-1}}_{\Delta k} &= 10 + \underbrace{0.5 \cdot k_{t-1} - k_{t-1}}_{=-0.5 \cdot k_{t-1}} && \text{(Difference Eq.)} \end{aligned}$$

$$\underbrace{\frac{\partial k_t}{\partial t}}_{\Delta k} = 10 - 0.5 \cdot k_t \qquad \text{(Differential Eq.)}$$

Dynamics and how to model it (cont.)

- Dynamics can be modelled under three major ways:
    - Difference equations / differential equations
    - Partial differential equations (PDE)
- **Examples:** take a, c, α as parameters

$$\underbrace{k_t - k_{t-1}}_{\Delta k} = a \cdot k_{t-1} - c \cdot k_{t-1}^{\alpha} \qquad \text{(Difference Eq.)}$$

$$\underbrace{\frac{\partial k_t}{\partial t}}_{\Delta k} = a \cdot k_t - c \cdot k_t^{\alpha} \qquad \text{(Differential Eq.)}$$

$$\underbrace{\frac{\partial k_t}{\partial t}}_{\Delta k} = a - c \cdot \frac{\partial k_t}{\partial u_t} \qquad \text{(Partial Diff. Eq.)}$$

- **Attention:** most difference/differential eq. can be easily solved; PDEs are very hard to solve
- We will **only deal with difference equations** in this course

Major ingredients of a dynamic model

- The **initial condition**
- The **transitional dynamics** process
- The steady state, or the **long term equilibrium** (or the fixed point)

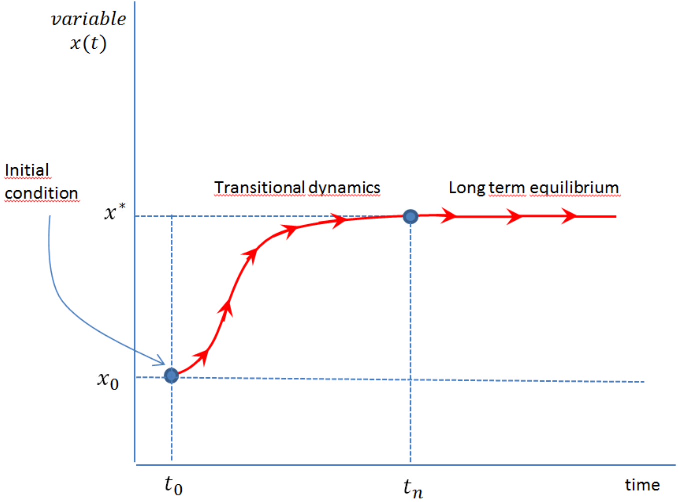
Definition

Long term equilibrium (LTE). This equilibrium is defined as the state in which the endogenous variables grow at a constant rate — which can be positive, negative or equal to zero — and time is allowed to proceed without bounds (time can go from 0 to ∞).

Definition

Transitional dynamics. This process represents the adjustment of the endogenous variables between the initial condition and the long term equilibrium — or between two long term equilibria if there is a change in some exogenous variable or parameter—. In this process, the endogenous variables can grow at either increasing or decreasing rates.

Major ingredients of a dynamic model: graphically



The three major questions about the LTE

- In the analyses of any kind of dynamic process there are three crucial questions:
  - Does the process converges to a LTE? Does it exist?
  - If it exists, is it stable or unstable?
  - If the LTE exists, it is unique or there are many equilibria?
- Answering the first is extremely easy ...
- ... but other two answers require much more formalism
- Let's see the answers

The condition imposed to calculate the LTE

- The first question:

  Does the process converges to a LTE? Does it exist?

- To answer this question, we have just to impose the following condition to our previous equations

$$\Delta x = 0$$

- For our difference equation ($\bar{x}$ is the LTE level of x_t) we have

$$x_{t+1} = x_t = \bar{x}$$

- Intuition:

$$\boxed{\begin{array}{lcl} x_{t+1} > x_t & \Longrightarrow & x_t \uparrow \\ x_{t+1} < x_t & \Longrightarrow & x_t \downarrow \\ x_{t+1} = x_t & \Longrightarrow & x_t = \bar{x} \text{ (constant)} \end{array}}$$

Stability and uniqueness of the LTE (cont.)

- Recall that the second and third questions are:
  - **If the LTE exists, is it stable or unstable?**
  - **If the LTE exists, is it unique or there are many equilibria?**
- To provide formal answers to these two questions would take us a long time
- So we will provide examples which will give intuition about the major points
- As you will see no major maths are required to understand the following points

Three linear examples

- Assume that our system can be represented by a rather simple equation
$$x_{t+1} = 10 + a \cdot x_t$$
- The answer to the three questions depend on parameter a :
$$\begin{aligned} |a| &< 1 \\ |a| &> 1 \\ |a| &= 1 \end{aligned}$$
- First case: $a = 0.5$.
- Second case: $a = 1.5$
- Third case: $a = 1$.

Our first case: a=0.5

- If we have

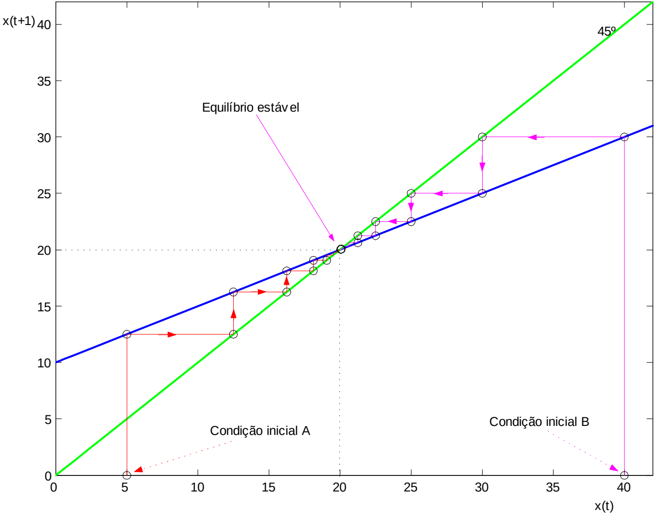
$$x_{t+1} = 10 + 0.5 \cdot x_t \tag{1}$$

- The LTE can be easily calculated

$$\begin{aligned} x_{t+1} &= x_t = \bar{x} \\ \bar{x} &= 10 + 0.5 \cdot \bar{x} \\ \bar{x} &= 20 \end{aligned}$$

- Answers:
  - The LTE exists: $\bar{x} = 20$
  - It is unique (see next figure)
  - It is stable: no matter what the initial conditions may be, it tends towards the LTE

LTE: exists, is unique and stable



Our second case: a=1.5

- Now, our system can be represented by

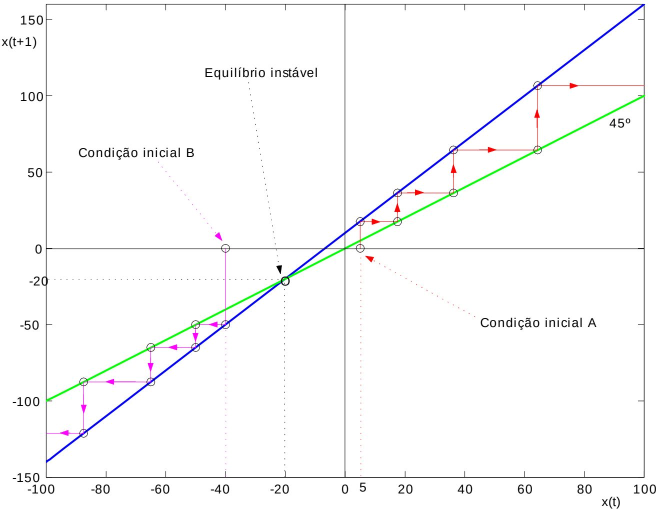
$$x_{t+1} = 10 + 1.5 \cdot x_t \tag{2}$$

- The LTE can be easily calculated

$$\begin{aligned} x_{t+1} &= x_t = \bar{x} \\ \bar{x} &= 10 + 1.5 \cdot \bar{x} \\ \bar{x} &= -20 \end{aligned}$$

- Answers:
  - The LTE exists: $\bar{x} = -20$
  - It is unique **(see next figure)**
  - It is unstable: the LTE only exists if the initial condition equals exactly the value of $\bar{x} = -20$. If there is a minor shock that forces the process to move away (even if only infinitesimally) from the LTE, the process diverges further and further away from the LTE.

LTE: exists, is unique but unstable



Our third case: a=1

- Now, our system can be represented by

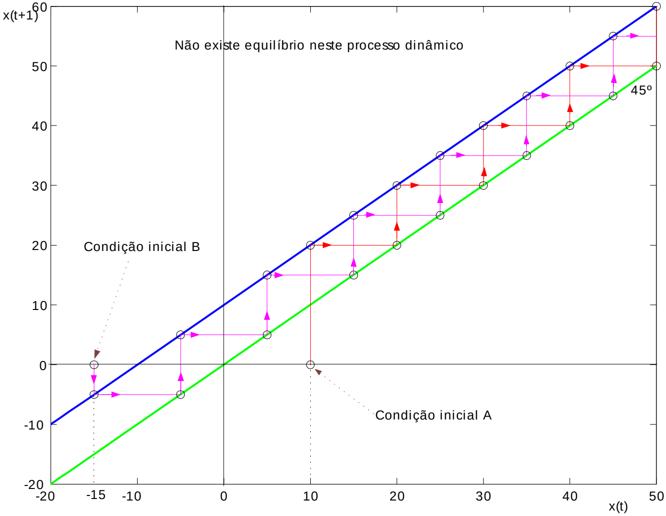
$$x_{t+1} = 10 + 1 \cdot x_t \tag{3}$$

- The LTE can be easily calculated

$$\begin{aligned} x_{t+1} &= x_t = \bar{x} \\ \bar{x} &= 10 + 1 \cdot \bar{x} \\ 0 \cdot \bar{x} &\neq 10 \end{aligned}$$

- Answers:
  - The LTE does not exists: no value of $\bar{x}$ satisfies $0 \cdot \bar{x} = 10$
  - Is it unique? Redundant answer **(see next figure)**
  - Is it stable? Redundant answer

LTE: it does not exist



A fourth example: multiple equilibria in a nonlinear case

- Consider that a process can be modelled by the nonlinear difference equation

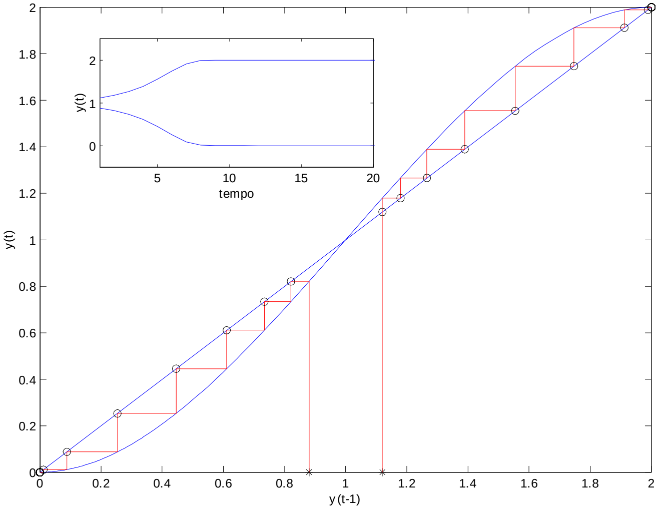
$$y_t = \frac{a_1 y_{t-1}^3 - a_2 y_{t-1}^2}{2} \qquad (4)$$

- Assume that the parameters are: $a_1 = -1$, $a_2 = 3$. The LTE is calculated as

$$y_t = y_{t-1} = \bar{y}$$

- Answers:
    - The LTE exists? YES! The three roots from that equation are : $\bar{y}_A = 0, \bar{y}_B = 1, \bar{y}_C = 2$
    - Is it unique? NO!! We have three values that satisfy the LTE condition **(see next figure)**
    - Is it unstable:we get two stable LTE and one unstable $(\bar{y}_B = 1)$

LTE: exists, not unique, some stable, some unstable



A recipe for tragedy: short term mixed up with long term

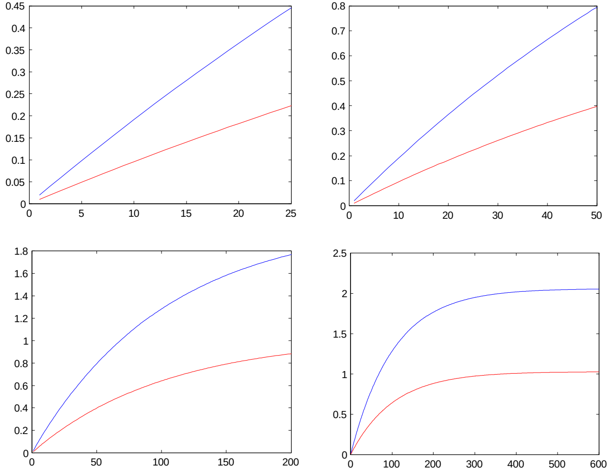
- Please, **never mix up** short term (transitional dynamics) with long term phenomena (LTE)
- A simple example. The evolution of the ratio of public debt to real GDP (z_t)

$$z_t = \psi + \tau \cdot z_{t-1} \quad , \qquad t = 0, 1, 2, ... \tag{5}$$

- ψ is the primary deficit (in % points); $\tau = \frac{1+r}{1+g}$, with g as the growth rate of real GDP, and r as the real interest rate
- Assume $g = 3\%, r = 2\%$, and two scenarios: $\psi_A = 0.01$, $\psi_B = 0.02$.
- What happens to the evolution of z_t?
- Answer: it depends upon the the period we choose in our simulations **(see next figure)**

Do not confuse short term with long term

Scenario A: red line; Scenario B: blue line

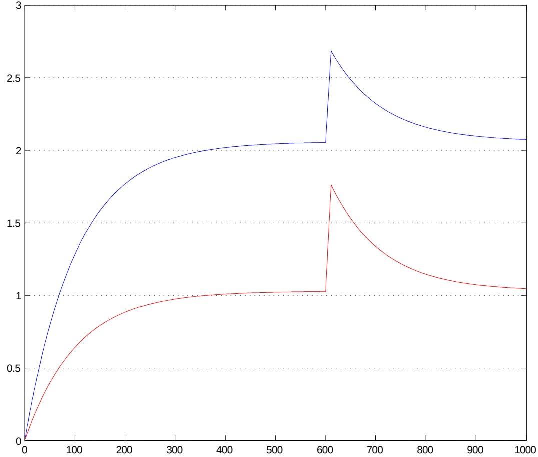


Do not confuse short term with long term (cont.)

1. For short periods of time (say, 25, 50 years), it looks like z_t grows without bound (explosive behavior)
2. But if we choose 600 years, we get a completely different result: z_t converges to a long term equilibrium in each scenario
3. Let's impose a shock. What happens if, between $t = 600, t = 610$, there is a shock to primary deficits in both scenarios
    1. $\psi = 0.08$ at $t = [600, 610]$
    2. ψ goes back again to normality after $t = 610$; (again $\psi_A = 0.01, \psi_B = 0.02$)
4. Looking at next figure what would you conclude?

Do not confuse sort term with long term (cont.)

Scenario A: red line; Scenario B: blue line



II – Major characteristics of a stochastic dynamic process

Stochastic dynamic processes

- A stochastic dynamic process has two major components:
  - A deterministic part
  - A random component or a random term
- **Example with no trend:**

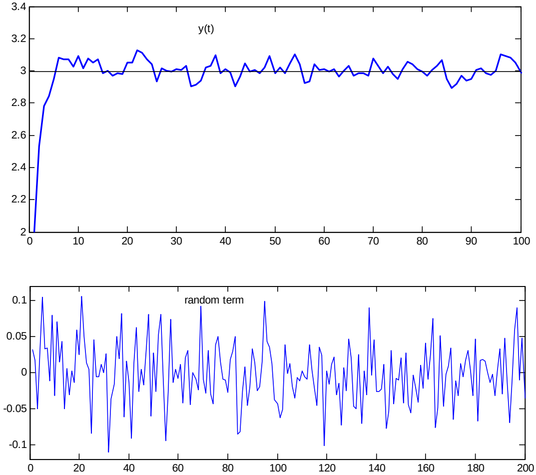
$$y_t = 1.5 + 0.5 \cdot y_{t-1} + \frac{1}{25}\varepsilon_t \qquad \text{(Stochastic difference Eq.)}$$

- ε_t is a **normal** random variable with independent and identically distributed observations

$$\varepsilon_t \sim iid(0, 1).$$

- The random term is given by: $\frac{1}{25}\varepsilon_t$
- The deterministic part given by: $1.5 + 0.5 \cdot y_{t-1}$

A stochastic dynamic process with no trend



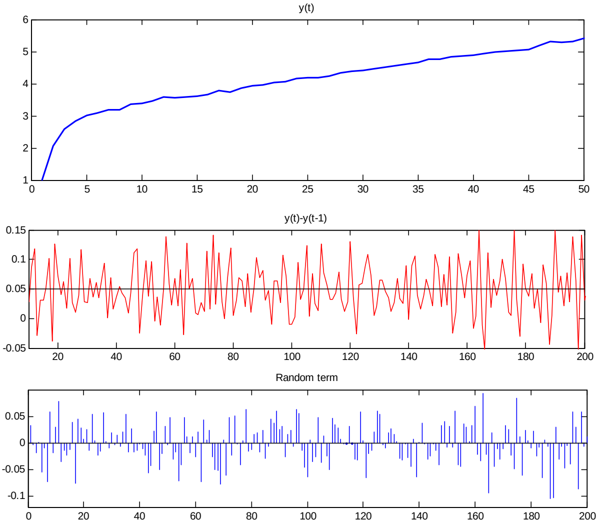
A stochastic dynamic process with a trend

1. Let's introduce a **time trend** into a stochastic process

$$y_t = a_0 + a_1 y_{t-1} + a_2 t + (1/25)\varepsilon_t$$

2. The novelty here is the time trend element: $a_2 \times t, \quad t = 0, 1, 2....$
3. The parameters assume these values: $a_0 = 1.5$, $a_1 = 0.5$, $a_2 = 0.025$, $\varepsilon_t \sim iid(0, 1)$
4. Let's represent three main results from this process:
   1. The original process (y_t)
   2. The first difference $(y_t - y_{t-1})$
   3. The random term:$(1/25)\varepsilon_t$
5. See next figure

A stochastic dynamic process with a trend (cont.)



A stochastic dynamic process with a trend: exercise

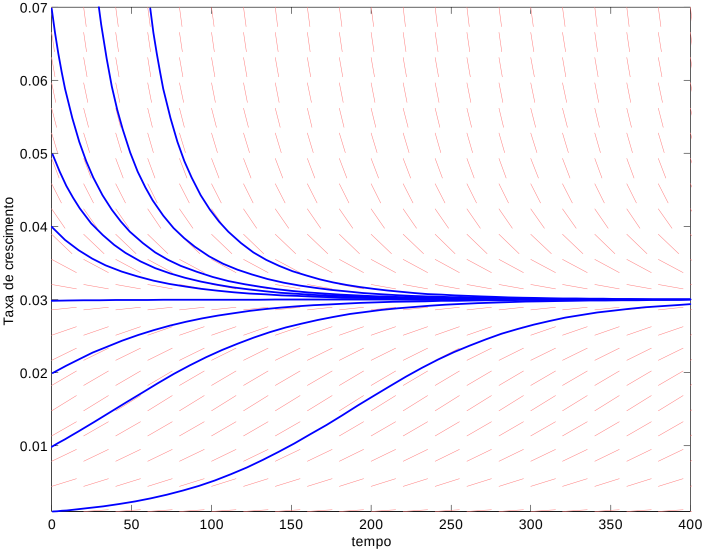
1. In the same process

$$y_t = a_0 + a_1 y_{t-1} + a_2 t + (1/25)\varepsilon_t$$

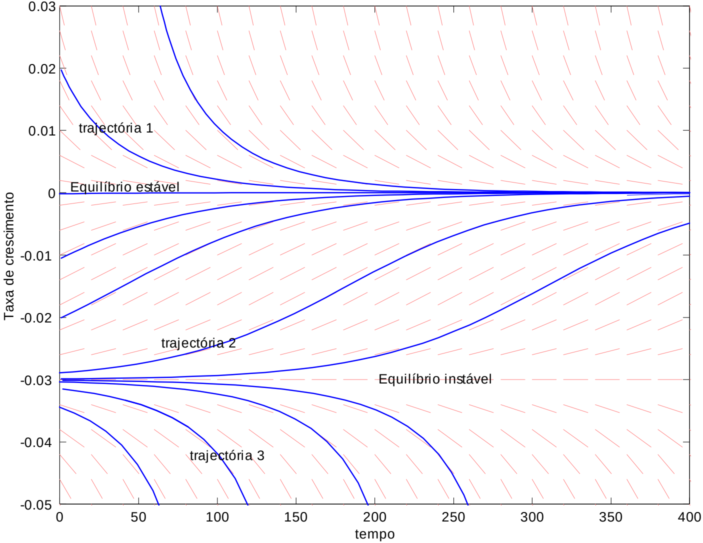
2. with the same parameters: $a_0 = 1.5$, $a_1 = 0.5$, $a_2 = 0.025$, $\varepsilon_t \sim iid(0, 1)$
3. Calculate the long term equilibrium value of the first difference: $(y_t - y_{t-1})$.
4. In the previous figure we may get some information about this.
5. It seems that the mean of $y_t - y_{t-1}$ is equal to 0.05. How can we get to this number?
6. Does the type of information in the previous figure makes you remember some thing that you learned in previous classes (stylized facts of the business cycles)?

III – Some examples of different types of dynamics

One stable equilibrium

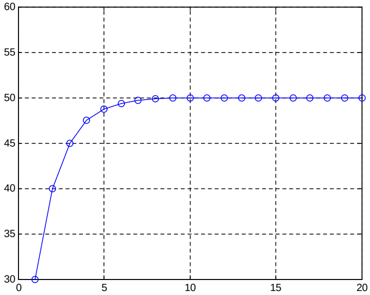


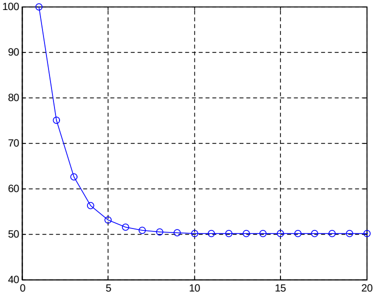
Multiple equilibria: one stable, one unstable



Monotonic convergence to a fixed point: a=0.5

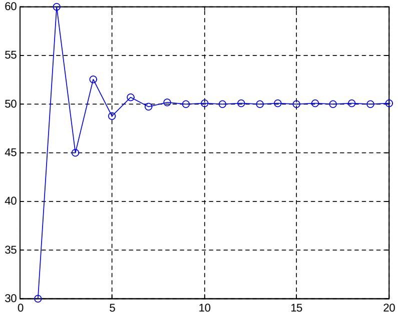
1. $y_t = 25 + 0.5 y_{t-1}$.
2. Left panel: $y(0) = 30$; Right panel $y(0) = 100$

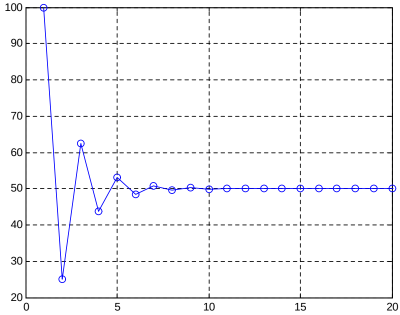




Oscillatory convergence to a fixed point: a=-0.5

1. $y_t = 75 - 0.5y_{t-1}$.
2. Left panel: $y(0) = 30$; Right panel $y(0) = 100$





IV — 2-Dimensional dynamic processes

Presenting the problem

1. **Until now** we have been dealing with a dynamic process that was represented by just **one equation** of motion (a difference equation)
2. **Now**, let's introduce a dynamic process that is described by a system of 2 (or more) coupled difference equations
3. Imagine something like this

$$\begin{aligned} x_{t+1} &= 10 + a_1 x_t + a_2 y_t \\ y_{t+1} &= 5 + a_3 x_t + a_4 y_t \end{aligned}$$

   where a_1, a_2, a_3, a_4 are parameters
4. In order to answer the standard questions about the dynamics, what should we do?
5. Can we just look at the values of the parameters individually and conclude about the three questions? **No**.

Answering the three questions

1. Does a LTE exist? If it exists, is it unique or there are many equilibria? If it exists, is it stable or unstable?
2. The answer to the two first questions follows similar reasoning as before:
    1. A LTE exists if

$$\begin{aligned} x_{t+1} &= x_t = \bar{x} \\ y_{t+1} &= y_t = \bar{y} \end{aligned}$$

    2. It is unique if there is only one pair $(\bar{x}, \bar{y})$ that satisfies both conditions above
3. However, the answer to the third question (about stability):
    1. does not depend just upon the values of parameters a_1, a_3
    2. It also depends upon parameter a_2, a_4
4. But as you will see, the logic is the same. We need **matrices** here.

Matrices to provide an answer about stability

1. The system can be written as follows

$$\begin{aligned} x_{t+1} &= 10 + a_1 x_t + a_2 y_t \\ y_{t+1} &= 5 + a_3 x_t + a_4 y_t \end{aligned}$$

2. Or in matricial form

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} \cdot \underbrace{\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix}}_{Z_{t+1}} = \underbrace{\begin{bmatrix} 10 \\ 5 \end{bmatrix}}_{B} + \underbrace{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_t \\ y_t \end{bmatrix}}_{Z_t}$$

3. This notation can be even further simplified as

$$I \cdot Z_{t+1} = B + A \cdot Z_t$$

4. Now, stability depends upon what happens to the **two eigenvalues** of matrix A. Let's call them λ_1, λ_2.

Similarity between dimension 1 and dimension 2

1. Notice how similar the two cases are in the end: a system with one single difference equation and with two (or more) equations
2. In the 2-Dimension case

$$I \cdot Z_{t+1} = B + A \cdot Z_t$$

3. The fixed point is obtained by imposing $Z_{t+1} = Z_t = \bar{Z}$, and get

$$\begin{aligned}(I - A)\bar{Z} &= B \\ \bar{Z} &= (I - A)^{-1}B\end{aligned}$$

4. In the 1-Dimension case, $x_{t+1} = b + ax_t$, by imposing $x_{t+1} = x_t = \bar{x}$, we got

$$\bar{x} = \frac{b}{1-a} = (1-a)^{-1}b$$

5. In 1-D, stability depended on the values of a. In the 2-D case, stability depends on the values (eigenvalues) of $A : (\lambda_1, \lambda_2)$.

Eigenvalues

1. We saw that
$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$
2. To calculate the two eigenvalues of A we write
$$[A - \lambda I_2] = \begin{bmatrix} a_1 - \lambda & a_2 \\ a_3 & a_4 - \lambda \end{bmatrix}$$
where I_2 is the identity matrix of order two.
3. Now, calculate the determinant of $(A - \lambda I_2)$ and make it equal to zero
$$\det\left[(a_1 - \lambda)(a_4 - \lambda) - (a_2 \times a_3)\right] = 0$$
4. We get
$$\lambda^2 - (a_1 + a_4)\lambda + (a_1 a_4 - a_2 a_3) = 0$$

Eigenvalues (cont.)

1. We obtained the following quadratic expression

$$\lambda^2 - (a_1 + a_4)\lambda + (a_1 a_4 - a_2 a_3) = 0$$

2. The two roots of a general quadratic equation like $a\lambda^2 - b\lambda + c = 0$ are given by

$$\lambda_1, \lambda_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. Applying this to our particular case we get

$$\lambda_1, \lambda_2 = \frac{(a_1 + a_4) \pm \sqrt{(a_1 + a_4)^2 - 4\,(a_1 a_4 - a_2 a_3)}}{2}$$

4. Now, there is a large number of possibilities:
   1. Both λ_i real numbers and $|\lambda_i| < 1$: fixed point stable
   2. Both λ_i real numbers and $|\lambda_i| > 1$: fixed point unstable
   3. Both λ_i real and $|\lambda_1| > 1, |\lambda_2| < 1$: fixed point is saddle unstable

Eigenvalues: complex numbers

1. There is still one fourth possibility: the eigenvalues are not real numbers, instead they are **complex numbers**
2. This occurs whenever

$$\underbrace{(a_1 + a_4)}_{trA}{}^2 < 4 \times \underbrace{(a_1 a_4 - a_2 a_3)}_{\det A}$$

3. If both λ_i are complex numbers: $\lambda_i = \alpha \pm \beta i$, $i = \sqrt{-1}$, then
   1. If $\sqrt{\alpha^2 + \beta^2} < 1$, through a **spiral**, the system converges to the fixed point
   2. If $\sqrt{\alpha^2 + \beta^2} > 1$, through a **spiral**, the system moves diverges from the fixed point
   3. If $\sqrt{\alpha^2 + \beta^2} = 1$, the system cycles around the fixed point

Eigenvalues: summary

1. Now we can summarize the results so far obtained.
2. Both λ_i are **real numbers**
   1. $|\lambda_i| < 1$: fixed point is stable
   2. $|\lambda_i| > 1$: fixed point unstable
   3. $|\lambda_1| > 1, |\lambda_2| < 1$: fixed point is saddle unstable
3. If both λ_i are **complex numbers**: $\lambda_i = \alpha \pm \beta i$, $i = \sqrt{-1}$, then
   1. If $\sqrt{\alpha^2 + \beta^2} < 1$, through a **spiral**, the system converges to the fixed point
   2. If $\sqrt{\alpha^2 + \beta^2} > 1$, through a **spiral**, the system moves diverges from the fixed point
   3. If $\sqrt{\alpha^2 + \beta^2} = 1$, the system cycles around the fixed point
4. Monotonic versus oscillatory convergence/divergence:
   1. $\lambda_i > 0$: monotonic convergence/divergence
   2. $\lambda_i < 0$: oscillatory convergence/divergence

Some exercises

1. Consider the following systems. Calculate the fixed points and conclude about the kind of stability in each one.

1

$$x_{t+1} = 10 + 1.1x_t - 0.4y_t$$
$$y_{t+1} = -25 + 0.5x_t + 0.2y_t$$

2

$$x_{t+1} = -10 + 1.2x_t + 0.8y_t$$
$$y_{t+1} = -50 + 0.5x_t + 0.2y_t$$

3

$$x_{t+1} = 2 + 0.8x_t + 1.4y_t$$
$$y_{t+1} = 15 + 0.4y_t$$

4

$$x_{t+1} = -10 + 1.2x_t - 0.8y_t$$
$$y_{t+1} = -50 + 0.5x_t + 1.5y_t$$

5

$$x_{t+1} = -10 + 1.2x_t - 0.8y_t$$
$$y_{t+1} = -50 + 0.5x_t + 0.2y_t$$

V– Nonlinear dynamic processes

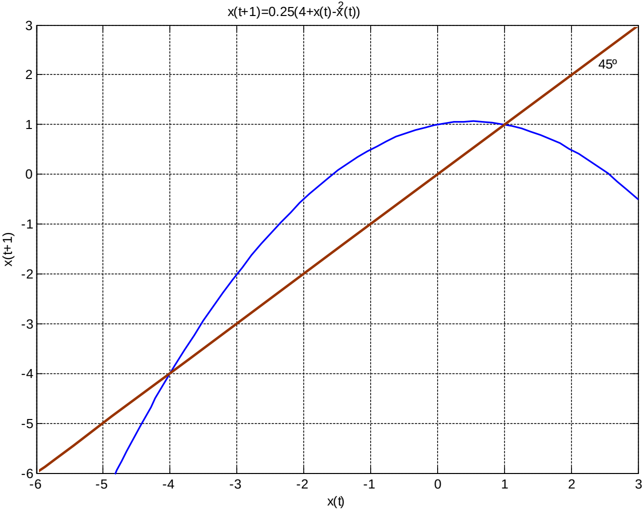
Everything is nonlinear in modern macro

1. Until now, we have only discussed dynamic processes that **were linear**
2. For the sake of simplicity, we kept this assumption hidden.
3. But in modern macroeconomics, almost all models have some form of **nonlinearity** in their structure
4. How can we come out with answers to the typical "three questions" in the case of nonlinearities?
5. The answer is: **linearize in the neighborhood of the fixed point.**
6. In most cases nothing is lost with this procedure. Let´s see how this is done.

An example

1. Suppose we have a nonlinear difference equation
$$x_{t+1} = f(x_t)$$
2. One example is
$$x_{t+1} = \frac{1}{4}(4 + x_t - x_t^2)$$
3. See the two fixed points $(-4, 1)$ in the next figure.
4. Are they stable or unstable?
5. From the linear analysis, we know that if $|f'(x_t)| > 1$, a fixed point is unstable, and stable if $|f'(x_t)| < 1$.
6. The problem is now that $f'(x_t)$ is not constant across the domain of x_t.

A nonlinear example (cont.)



Can we trust in our eyes?

1. We could try one trick:
   1. see whether the slope of the tangent to each fixed point is lower, or higher than one.
   2. Like in the **next figure.**
2. The problem is that it is very dangerous to base our conclusions merely upon what our eyes may observe
3. See the following example:

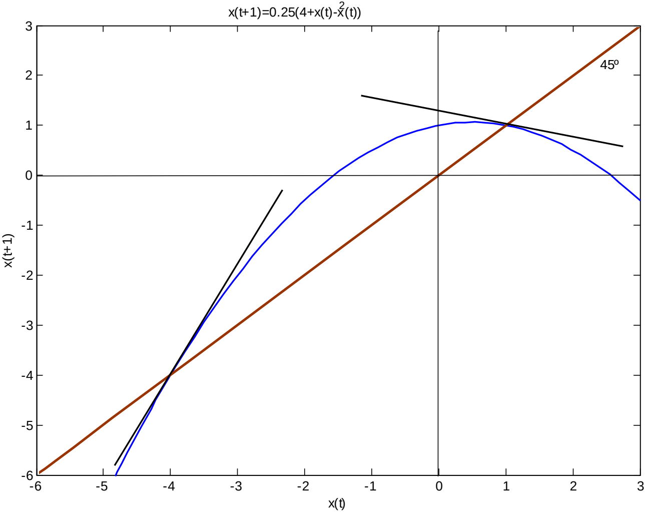
$$x_{t+1} = ax_t(1 - x_t)$$

4. Let's try four different values for a :

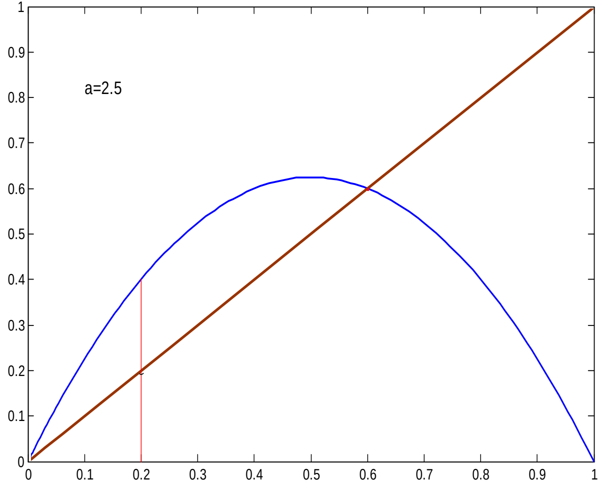
$$a = 2.5 \ , \quad a = 2.8 \ , \quad a = 3.2 \ , \quad a = 3.8$$

5. Observe the following figures describing each case.
6. Can you conclude about the particular type of dynamics just by looking at each figure?

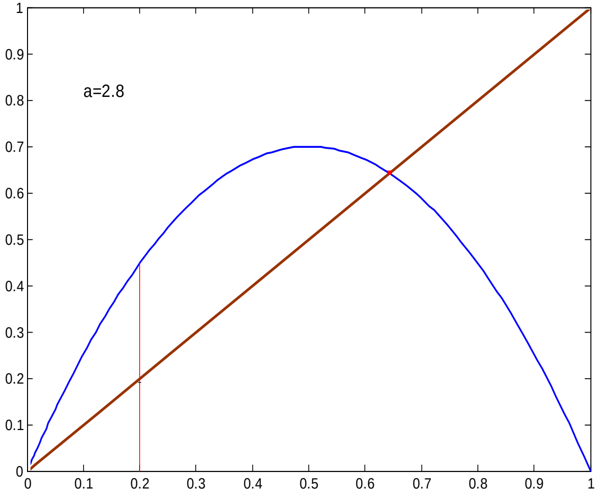
An nonlinear example: a tangent at the fixed points



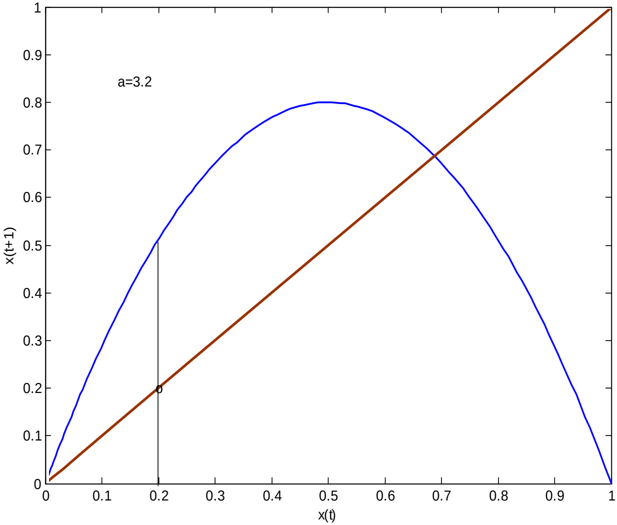
Try your eyes (a=2.5)



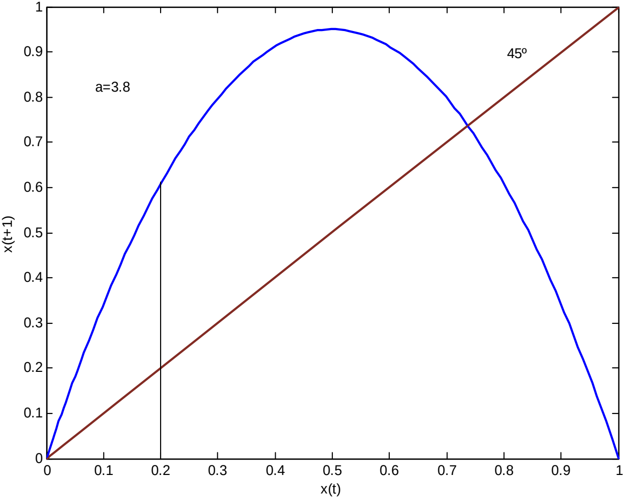
Try your eyes (a=2.8)



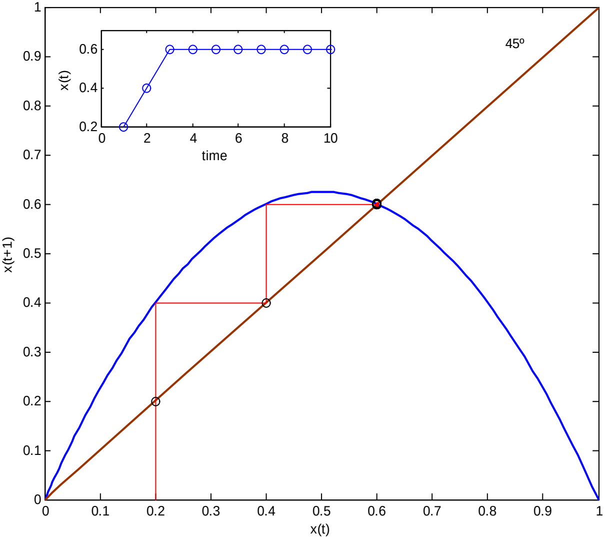
Try your eyes (a=3.2)



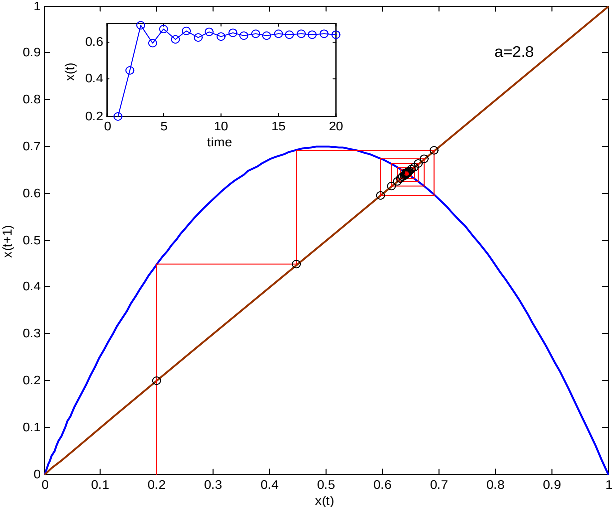
Try your eyes (a=3.8)



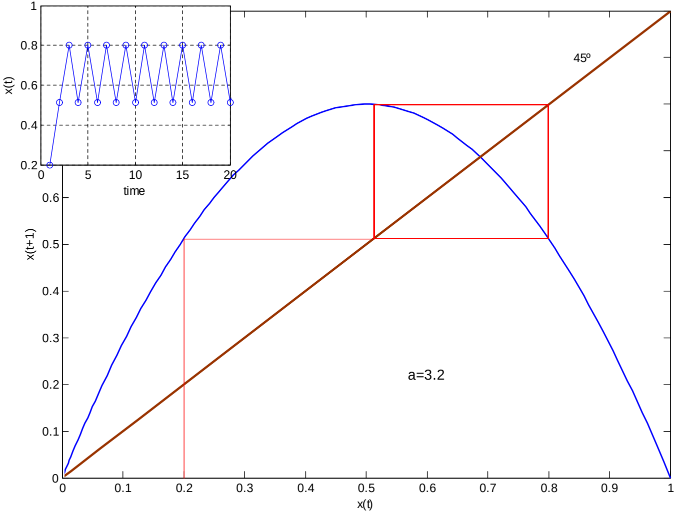
Check yours eyes: a=2.5



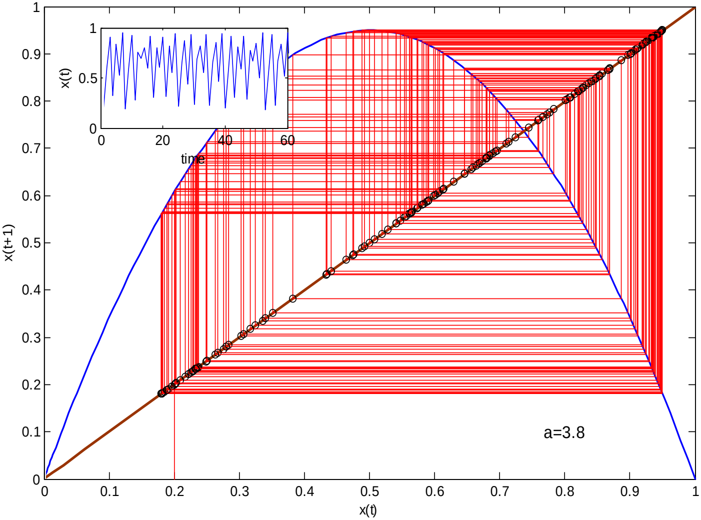
Check yours eyes: a=2.8



Check yours eyes: a=3.2



Check yours eyes: a=3.8



VI – Linearization (or linear approximation) of nonlinear processes

For 1 dimensional nonlinear systems

1. Let's go back to our example where $x_{t+1} = f(x_t)$:

$$x_{t+1} = f(x_t) = \frac{1}{4}(4 + x_t - x_t^2)$$

2. We want to approximate f near a point x_t (a fixed point here: $\bar{x}$)
3. This is obtained by applying the following expression

$$f(x_t) \approx f'(\bar{x})(x_t - \bar{x}) + f(\bar{x})$$

4. Let's do it. Firstly, calculate

$$f'(x_t) = \frac{(1 - 2x_t)}{4}, \quad f'(\bar{x} = 1) = -\frac{1}{4}, \quad f'(\bar{x} = -4) = \frac{9}{4}$$

5. Now at each fixed point we have the following approximations

$$\begin{aligned} f(x_t)_{\bar{x}=1} &= -(1/4)x_t + (5/4) \\ f(x_t)_{\bar{x}=-4} &= (9/4)x_t + 5 \end{aligned}$$

6. It's easy to see what kind of stability we have in each fixed point. Can you answer that?

For 2 dimensional nonlinear systems

1. Suppose you have a nonlinear system like this one

$$\begin{aligned} x_{t+1} &= g(x_t, y_t) = -4 + x_t^2 + y_t^2 \\ y_{t+1} &= h(x_t, y_t) = 0.5 + 0.5x_t + y_t \end{aligned}$$

2. This may be written in matricial form

$$z_{t+1} = f(z_t)$$

3. The linear approximation works exactly like in the 1D case

$$f(z_t) \approx f'(\bar{z})(z_t - \bar{z}) + f(\bar{z})$$

4. The only difference now is that $f'(\bar{z})$ is a $n \times n$ matrix (not a scalar)

$$f'(\bar{z}) = \begin{bmatrix} \frac{\partial g(\bar{z})}{\partial x} & \frac{\partial g(\bar{z})}{\partial y} \\ \frac{\partial h(\bar{z})}{\partial x} & \frac{\partial h(\bar{z})}{\partial y} \end{bmatrix}$$

5. Finally, at each fixed point calculate the eigenvalues of $f'(\bar{z})$, and conclude accordingly to what you learned in the linear case.

Solving the 2 dimensional nonlinear example

1. Suppose you have a nonlinear system like this one

$$\begin{aligned} x_{t+1} &= g(x_t, y_t) = -0.09 + x_t^2 + y_t^2 \\ y_{t+1} &= h(x_t, y_t) = -0.5 + 0.5x_t + y_t \end{aligned}$$

2. The fixed point is determined by imposing the conditions

$$\begin{aligned} x_{t+1} &= x_t = \bar{x} \\ y_{t+1} &= y_t = \bar{y} \end{aligned}$$

3. From which we can get two fixed points

$$\begin{aligned} \bar{z}_1 &= (\bar{x} = 1, \ \ \bar{y} = 0.3) \\ \bar{z}_2 &= (\bar{x} = 1, \ \ \bar{y} = -0.3) \end{aligned}$$

4. Next, calculate $f'(\bar{z})$ for each fixed point

Solving the 2 dimensional nonlinear example (cont.)

1. Calculating $f'(\bar{z})$ for each fixed point
2. For fixed point $\bar{z}_1 = (\bar{x} = 1, \bar{y} = 0.3)$ we get

$$f'(\bar{z}) = \begin{bmatrix} \frac{\partial g(\bar{z})}{\partial x} = 2x = 2 & \frac{\partial g(\bar{z})}{\partial y} = 2y = 0.6 \\ \frac{\partial h(\bar{z})}{\partial x} = 0.5 & \frac{\partial h(\bar{z})}{\partial y} = 1 \end{bmatrix}$$

3. For point $\bar{z}_2 = (\bar{x} = 1, \bar{y} = -0.3)$ we get

$$f'(\bar{z}) = \begin{bmatrix} \frac{\partial g(\bar{z})}{\partial x} = 2x = 2 & \frac{\partial g(\bar{z})}{\partial y} = 2y = -0.6 \\ \frac{\partial h(\bar{z})}{\partial x} = 0.5 & \frac{\partial h(\bar{z})}{\partial y} = 1 \end{bmatrix}$$

4. Finally, at each $\bar{z}_i$ calculate the eigenvalues of $f'(\bar{z})$, and conclude accordingly to what you learned in the linear case.

$$\bar{z}_1 \quad : \quad \lambda_1 = 2.2, \quad \lambda_2 = 0.75$$

$$\bar{z}_2 \quad : \quad \lambda_1 = 1.5 + 0.223i, \quad \lambda_2 = 1.5 - 0.223i, \quad \sqrt{1.5^2 + 0.223^2} = 1.51$$

VII – What have we learned?

What have we learned?

1. In economics, we need to know how to deal with dynamics: any economic process evolves in time, so ...
2. Answer the three fundamental questions: (i) does the economic process has a long term equilibrium (LTE)? Is it stable or unstable? Is it unique, or are there multiple equilibria?
3. You should know how to deal with a **linear deterministic** difference equation like:

$$x_{t+1} = 10 + a \cdot x_t, \quad a \text{ is a parameter}$$

4. You should know how to deal with a **linear stochastic** difference equation like:

$$x_{t+1} = 10 + a \cdot x_t + \varepsilon_t \quad , \quad \varepsilon_t \tilde{} iid(0,1)$$

5. You should know how to deal with a linear stochastic difference equation **with a trend** like:

$$x_{t+1} = 10 + a \cdot x_t + b \cdot t + \varepsilon_t \quad , \quad \varepsilon_t \tilde{} iid(0,1), \ t = 0, 1, 2 ...$$

What have we learned? (cont.)

1. You also need to have some basic knowledge of **matrices** in order to deal with a **system of linear** difference equations like

$$\begin{aligned} x_{t+1} &= 10 + 1.1x_t - 0.4y_t \\ y_{t+1} &= -25 + 0.5x_t + 0.2y_t \end{aligned}$$

2. You need to know the concept of **linear approximation (linearization),** to deal with a **nonlinear** difference equation like:

$$x_{t+1} = f(x_t) = \frac{1}{4}(4 + x_t - x_t^2)$$

3. You should know how to deal with a **system of nonlinear** difference equations like:

$$\begin{aligned} x_{t+1} &= g(x_t, y_t) = -0.09 + x_t^2 + y_t^2 \\ y_{t+1} &= h(x_t, y_t) = -0.5 + 0.5x_t + y_t \end{aligned}$$

Reading

Reading

1. Points I,II and III are expected to be just some revisions of material that you have already covered in introductory maths courses. So, just use as much as possible the slides in your study for these points.
2. For points IV to VI, if you feel like needing some further support from a textbook in order to consolidate your knowledge, please read the corresponding parts of Chapters 2 and 3 of Edward R. Scheinerman (1996). Invitation to Dynamical Systems, Prentice Hall.
   1. But notice that we cover only discrete time problems; continuous time problems is not part of our course. So do not worry about continuous time in those two chapters.

Answers to the 5 problems above

Answers to problem 1

Equilibrium exists and is stable

$$\begin{aligned} x &= 10 + 1.1x - 0.4y \\ y &= -25 + 0.5x + 0.2y \end{aligned}$$

Solution is: $\{[x = 150.0, y = 62.5]\}$,

$$A = \begin{bmatrix} 1.1 & -0.4 \\ 0.5 & 0.2 \end{bmatrix}$$

eigenvalues: $0.7, 0.6$
trace: 1.3,
determinant: 0.42

Answers to problem 2

Equilibrium exists and is saddle unstable

$$\begin{aligned} x &= -10 + 1.2x + 0.8y \\ y &= -50 + 0.5x + 0.2y \end{aligned}$$

, Solution is: $\{[x = 133.33, y = 20.833]\}$,

$A = \begin{bmatrix} 1.2 & 0.8 \\ 0.5 & 0.2 \end{bmatrix}$,

eigenvalues: $1.5062, -0.10623$
trace: 1.4,
determinant: -0.16

Answers to problem 3

Equilibrium exists and is stable

$$\begin{aligned} x &= 2 + 0.8x + 1.4y \\ y &= 15 + 0x + 0.4y \end{aligned}$$

, Solution is: $\{[x = 185.0, y = 25.0]\}$,

$$A = \begin{bmatrix} 0.8 & 1.4 \\ 0 & 0.4 \end{bmatrix}$$

eigenvalues: $0.8, 0.4$
trace: 1.2,
determinant: 0.32

Answers to problem 4

Equilibrium exists, but diverges from the fixed point through a spiral $\left(\sqrt{\alpha^2+\beta^2}>1\right)$

$$\begin{aligned} x &= -10+1.2x-0.8y \\ y &= -50+0.5x+1.5y \end{aligned}$$

, Solution is: $\{[x=90.0, y=10.0]\}$
$A=\begin{bmatrix} 1.2 & -0.8 \\ 0.5 & 1.5 \end{bmatrix}$,

eigenvalues: $1.35+0.61441i, 1.35-0.61441i$
trace: 2.7,
determinant: 2.2
$\sqrt{(1.35)^2+(0.61441)^2}=1.4832>1$

Answers to problem 5

Equilibrium exists, and converges to the fixed point through a spiral $\left(\sqrt{\alpha^2 + \beta^2} < 1\right)$

$$\begin{aligned} x &= -10 + 1.2x - 0.8y \\ y &= -50 + 0.5x + 0.2y \end{aligned}$$

, Solution is: $\{[x = 133.33, y = 20.833]\}$,
$A = \begin{bmatrix} 1.2 & -0.8 \\ 0.5 & 0.2 \end{bmatrix}$

eigenvalues: $0.7 + 0.38730i, 0.7 - 0.38730i$
trace: 1.4,
determinant: 0.64
$\sqrt{(0.7)^2 + (0.3873)^2} = 0.8 < 1$