Optimal Monetary Policy in the NKM

Vivaldo M. Mendes

ISCTE — Lisbon University Institute

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Summary

1. Recap the Poole's model: Interest rates vs Money supply: what is best?
2. Optimal monetary policy: discretion
3. Optimal monetary policy: commitment to simple rules
4. Optimal monetary policy: optimal commitment
5. Stability and Indeterminacy

I — Interest rates vs Money supply: what is best?

Interest rate or money supply as operating target?

1. In 1970, William Poole demonstrated a fundamental result:[1]
    1. Interest rate is a better instrument than monetary base if the **variance of money demand shocks** is larger than the **variance of aggregate demand shocks**
    2. This has been easily confirmed by empirical evidence
2. In his analysis the Central Bank's objective was to minimize fluctuations in the output or deviations from the natural level of output (similar to the NKM)
3. Gerry Bouey, a former governor of Canada's Central Bank:
    1. *"We didn't abandon the monetary aggregates they abandoned us"*
4. **Since 1989**, the Fed has used fed funds rates as instruments (operating targets) to hit its intermediate target of short term interest rates.

[1] William Poole (1970). "Optimal choice of monetary policy instruments in a simple stochastic macro model," Staff Studies 57, Federal Reserve Board, Washington.

Interest rate as the operating target

1. Until now, we have 3 endogenous variables $\{x_{t+s}, \pi_{t+s}, r_{t+s}\}_{s=0}^{s=\infty}$, but only two equations (IS, AS or the New Phillips curve).
2. Top close the model, we need another equation:
   1. **Old view**: the CB controls the money supply, and the market determines r_t,
   2. **New view:** the central bank controls the r_t, and the market determines the level of money in the market.
3. **From now onwards,** we will stick to the new view: interest rate as the monetary operating target
4. We will look for an **interest rate** that satisfies optimality in monetary policy.

Interest rate rules: examples

1. In the new view, the central bank can determine r_t in two main different ways: in an ad-hoc manner, and in an optimal way
2. **Ad-hoc interest rate rules**. These are variants on the Taylor rule:
   1. An exogenous interest rate rule:
   $$r_t = r_t^n$$
   2. An interest rate rule with feedback from target variables
   $$r_t = r_t^n + \gamma_\pi \pi_t + \gamma_x x_t \ , \qquad \gamma_\pi, \gamma_x > 0$$
   3. A forward-looking interest rate rule
   $$r_t = r_t^n + \gamma_\pi E_t \pi_{t+1} + \gamma_x E_t x_{t+1} \ , \qquad \gamma_\pi, \gamma_x > 0$$
3. **Optimal interest rate rules**. These can be set under two alternative ways: "*with commitment*" or "*with discretion*"
4. In what follows we will concentrate on **optimal** interest rate rules

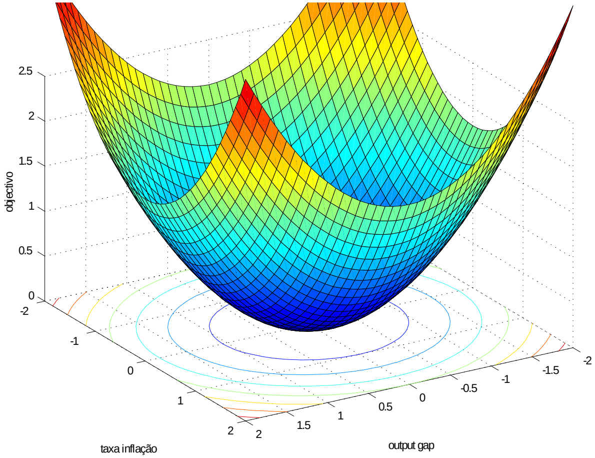
The Central Bank loss function

1. The central bank main objective

$$\min \underbrace{\frac{1}{2}E_t\left[\sum_{s=0}^{\infty}\beta^s\left(ax_{t+s}^2+\pi_{t+s}^2\right)\right]}_{\text{Loss function } (L)} \qquad (1)$$

2. with $a > 0$. This parameter shows whether the CB is more concerned with inflation than with output $(0 < a < 1)$, or the opposite $(a > 1)$
3. Therefore the problem of the central bank is to choose $\{x_t, \pi_t, r_t\}_{s=0}^{s=\infty}$, to min the L function
4. In which, the policy instrument is the short term nominal interest rate (r_t)
5. **Next figure** shows that the **unconstrained** values of x_t, π_t that min the L function (both zero).

The central bank loss function



Optimal interest rates: the problem

1. As we will see the **constrained** values of x_t, π_t that min the L function will depend on the types of exogenous shocks.
2. This is because we have a linear Rational Expectations model, and therefore the endogenous variables in the steady state only respond to shocks
3. Set the optimization process as

$$\min \frac{1}{2} E_t \left[\sum_{s=0}^{\infty} \beta^s \left(a x_{t+s}^2 + \pi_{t+s}^2 \right) \right] \quad \text{(Objective)}$$

subject to

$$x_t = -\varphi \left(r_t - E_t \pi_{t+1} \right) + E_t x_{t+1} + \mu_t \quad \text{(IS)}$$

$$\pi_t = \beta \cdot E_t \pi_{t+1} + \lambda x_t + v_t \quad \text{(AS)}$$

II — Optimal monetary policy: discretion

Three steps and a reminder

1. In order to arrive at the optimal level for short term interest rates under discretion we have to undertake three steps:
    1. **Deterministic part**: Solve the constrained intertemporal problem for the deterministic part of the model
    2. **Bring shocks in**: Then introduce those two shocks into the solution in (a)
    3. **Back again to the IS and AS functions**: use results in (b) to solve for the optimal interest rate
2. **With discretion** remember that the central bank:
    1. optimizes period by period
    2. takes the expectations of the private agents as given: $E_t\pi_{t+1}, E_t x_{t+1}$ as constants in the *min* problem

Optimal solution (ignoring shocks)

1. Given that expectations are given, the problem can be set in period t involving only the L and AS functions, because the problem is determinate as such[2]

$$\min \frac{1}{2}\left(ax_t^2 + \pi_t^2\right) = \min \frac{1}{2}[ax_t^2 + (\underbrace{\beta \cdot E_t\pi_{t+1} + \lambda x_t + v_t}_{=\pi_t})^2] \tag{2}$$

2. The FOC with respect to x_t, taking $E_t\pi_{t+1}$ as given, is as follows

$$\frac{\partial L}{\partial x_t} = 0 \Rightarrow ax_t + \lambda\left(\beta E_t\pi_{t+1} + \lambda x_t + v_t\right) = 0 \tag{3}$$

3. From eq.(3), we get a relationship between x_t, π_t that min L

$$\begin{aligned} ax_t + \lambda\pi_t &= 0 \\ x_t &= -\frac{\lambda}{a}\pi_t \end{aligned} \tag{4}$$

4. We have solved for the deterministic part of the model. But we need to know what happens to the model **with the shocks in**.

Bringing the shocks back

1. Insert eq. (4) back into the AS function and get

$$\begin{aligned}\pi_t &= \beta E_t \pi_{t+1} + \lambda \left(-\frac{\lambda}{a} \pi_t \right) + v_t \\ &= \beta E_t \pi_{t+1} - \frac{\lambda^2}{a} \pi_t + v_t \\ &= \frac{a\beta}{a+\lambda^2} E_t \pi_{t+1} + \frac{a}{a+\lambda^2} v_t \qquad (5)\end{aligned}$$

2. What kind of equation is (5)?

Solving for the inflation rate (with shocks back)

1. Eq. (5) is a linear Rational Expectations equation in π_t, which we already know how to solve, e.g., by using the method of repeated substitution.
2. In order to solve eq.(5), we need to remember that v_t is a supply shock
$$v_t = \rho_v v_{t-1} + \varepsilon_t,$$
with $\varepsilon_t \sim (0, \sigma_v^2)$, and $0 < \rho_v < 1$.
3. Two points should be recalled here:
$$E_t \varepsilon_{t+1} = 0 \quad , \quad E_t v_{t+1} = \rho_v v_t$$
4. So the solution to eq. (5) is (see Appendix 1 for details)
$$\pi_t = a\phi v_t$$
where, for simplicity, we defined $\phi = \dfrac{1}{a(1 - \beta\rho_v) + \lambda^2}$.

Solving for the output gap (with shocks back)

1. The optimal level of inflation was obtained as

$$\pi_t = a\phi v_t$$

2. With optimal π_t, we can get the optimal value for x_t by using eq. (4),
3. This is given by

$$\begin{aligned} x_t &= -\frac{\lambda}{a}\pi_t \\ &= -\lambda\phi v_t \end{aligned} \tag{6}$$

4. Now we can also obtain the optimal value for r_t by using the IS function and the optimal values for π_t and x_t, from (8) and (6:

$$IS : x_t = -\varphi\left(r_t - E_t\pi_{t+1}\right) + E_t x_{t+1} + \mu_t$$

5. In the IS function we have x_t, r_t and $E_t\pi_{t+1}, E_t x_{t+1}$, lets apply a trick

Solving for the optimal interest rate ... with a trick

1. First recall that
   1. As $v_t = \rho_v v_{t-1} + \varepsilon_t$, with $\varepsilon_t \sim (0, \sigma_v^2)$, then

$$E_t v_{t+1} = \rho_v v_t$$

   2. From eq.(8), $\pi_t = a\phi v_t$. Then

$$\begin{aligned} E_t \pi_{t+1} &= E_t\{a\phi v_{t+1}\} = a\phi E_t v_{t+1} \\ E_t \pi_{t+1} &= a\phi\rho_v v_t \end{aligned}$$

2. Let's substitute π_t, x_t back into the IS (Appendix 2 for details)

$$\begin{aligned} x_t &= -\varphi\left(r_t - E_t\pi_{t+1}\right) + E_t x_{t+1} + \mu_t \\ -\lambda\phi v_t &= -\varphi r_t + \varphi E_t\{a\phi v_{t+1}\} + E_t\{-\lambda\phi v_{t+1}\} + \mu_t \\ & .... \\ r_t &= \frac{1}{\varphi}\left(\frac{\lambda(1-\rho_v)}{a\rho_v} + \varphi\right) E_t\pi_{t+1} + \frac{1}{\varphi}\mu_t \end{aligned}$$

Discretion: summary of main results

1. From the previous exercise, we can write the optimal value for r_t as

$$r_t = \psi E_t \pi_{t+1} + \frac{1}{\varphi}\mu_t \tag{7}$$

   with $\psi = 1 + \left[\lambda(1-\rho_v)/\varphi a \rho_v\right]$. Notice that $\psi > 1$.
2. Now we can summarize the major results of the NKM
3. The optimal monetary policy with discretion can be characterized by (superscript d for discretion):

$$\begin{aligned} x_t^d &= -\lambda \phi v_t \\ \pi_t^d &= a \phi v_t \\ r_t^d &= \psi E_t \pi_{t+1} + \frac{1}{\varphi}\mu_t \end{aligned}$$

4. a, φ (parameters) and λ, ψ, ϕ are sum of parameters with $\phi > 0$, $0 < \lambda < 1$, $\psi > 1$.

Discretion: major result 1

$$
\begin{aligned}
x_t^d &= -\lambda\phi v_t \\
\pi_t^d &= a\phi v_t \\
r_t^d &= \psi E_t \pi_{t+1} + \frac{1}{\varphi}\mu_t
\end{aligned}
$$

1. **Result 1. Trade–off between inflation and output.** This trade-off only exists if there are supply side shocks (that is, if the variance of these shocks is different from zero).

Proof.

As it was assumed that $\varepsilon_t \sim (0, \sigma_\epsilon^2)$, then $E[\varepsilon_t] = 0$. If $\sigma_\epsilon^2 = 0$ we will have $v_t = 0, \ \forall t$. As $v_t = 0$, then $x_t^d = \pi_t^d = 0, \forall t$. That is, the actual output is always equal to potential output, and inflation is zero, in every period of time. □

Discretion: major result 2

1. **Result 2**. **"Inflation targeting"**. Optimal monetary policy requires "inflation targeting", which implies that the Central Bank (CB) should aim for convergence of inflation to its target value over the long term.

Proof.

As we know that $E_t \pi_{t+1} = a\phi\rho_v v_t$, then

$$\lim_{s\to\infty} E_t \pi^d_{t+s} = \lim_{s\to\infty} a\phi \left(\rho_v\right)^s v_t = 0$$

as $\rho_v < 1$. $\square$

Discretion: major result 3

1. **Result 3**. **CB's aggressive response**. Optimal monetary policy requires that, if there is an increase in expected inflation of a certain amount, this should be counterweighted by an increase in short term interest rates that is larger than the increase in expected inflation. This will lead to higher expected real interest rates, which in turn will constrain aggregate demand, and subsequently leading to lower expected inflation.

Proof.

We know that $r_t^d = \psi E_t \pi_{t+1}^d + \frac{1}{\varphi}\mu_t$, with $\psi = 1 + [\lambda(1-\rho_v)/\varphi a \rho_v] > 1$. Therefore $\partial r_t^d / \partial E_t \pi_{t+1}^d = \psi > 1$. □

Discretion: major result 4

1. **Result 4**. **Supply shocks: monetary policy neutrality**. Optimal monetary policy requires that the Central Bank should adjust short term nominal interest rates in order accommodate unexpected shock on the demand side of the economy (μ_t); however, monetary policy should remain unaltered if the shocks are on the supply side (v_t).

Proof.

From equation (7) we know that $r_t^d = \psi E_t \pi_{t+1} + \frac{1}{\varphi}\mu_t$, with $\varphi > 0$.
Therefore $dr_t^d/dv_t = 0, dr_t/d\mu_t > 0$. □

Appendix 1 – Solving a linear RE equation

Just apply repeated substitution and take into account that $E_t\varepsilon_{t+1} = 0$, $E_t v_{t+1} = \rho_v v_t$

$$\begin{aligned}
\pi_t &= \frac{a\beta}{a+\lambda^2} E_t\pi_{t+1} + \frac{a}{a+\lambda^2} v_t \\
\pi_t &= \frac{a\beta}{a+\lambda^2} E_t \left(\frac{a\beta}{a+\lambda^2} E_{t+1}\pi_{t+2} + \frac{a}{a+\lambda^2} v_{t+1} \right) + \frac{a}{a+\lambda^2} v_t \\
\pi_t &= \left(\frac{a\beta}{a+\lambda^2} \right)^2 E_t\pi_{t+2} + \frac{a}{a+\lambda^2} \left(v_t + \frac{a\beta}{a+\lambda^2} E_t v_{t+1} \right) \\
\pi_t &= \ldots \\
\pi_t &= a \frac{1}{a(1-\beta\rho_v)+\lambda^2} v_t \\
\pi_t &= a\phi v_t
\end{aligned} \tag{8}$$

where, for simplicity, we defined $\phi = \dfrac{1}{a(1-\beta\rho_v)+\lambda^2}$.

Appendix 2

Optimal levels of π_t, x_t into the IS

$$
\begin{aligned}
x_t &= -\varphi\left(r_t - E_t\pi_{t+1}\right) + E_t x_{t+1} + \mu_t \\
-\lambda\phi v_t &= -\varphi r_t + \varphi E_t\{a\phi v_{t+1}\} + E_t\{-\lambda\phi v_{t+1}\} + \mu_t \\
-\lambda\phi v_t &= -\varphi r_t + \varphi a\phi \underbrace{E_t\{v_{t+1}\}}_{=\rho_v v_t} - \lambda\phi \underbrace{E_t\{v_{t+1}\}}_{=\rho_v v_t} + \mu_t \\
\varphi r_t &= \lambda\phi v_t + \varphi a\rho_v\phi v_t - \rho_v\lambda\phi v_t + \mu_t \\
\varphi r_t &= \lambda\phi v_t \frac{a\rho_v}{a\rho_v} + \varphi\rho_v a\phi v_t - \lambda\rho_v\phi v_t \frac{a}{a} + \mu_t \\
\varphi r_t &= \left(\frac{\lambda(1-\rho_v)}{a\rho_v} + \varphi\right) \underbrace{\rho_v a\phi v_t}_{=E_t\pi_{t+1}} + \mu_t \\
r_t &= \frac{1}{\varphi}\left(\frac{\lambda(1-\rho_v)}{a\rho_v} + \varphi\right) E_t\pi_{t+1} + \frac{1}{\varphi}\mu_t
\end{aligned}
$$

III — Optimal monetary policy: commitment to simple rules

From here onwards, material not covered due to lack of time

Commitment

1. What happens if instead the central bank commits to a rule over time?
2. This can be done under two possibilities:
   1. **Global commitment**: the rule is chosen in an optimal way:

      what rule is the best one from all possible rules in order to min the Loss function?
   2. **Suboptimal commitment**: the rule is not chosen in an optimal way, is chosen as a simple rule

      but once chosen the central bank sticks to it.
3. We will illustrate both cases here
4. Let's start with commitment to a simple rule

Commitment: a simple rule

1. Assume that the central bank commits to the following simple rule

$$x_t = -\omega \cdot v_t \tag{9}$$

2. The fundamental point is to obtain the optimal value of ω , such that the L function is minimized
3. Notice the logic behind this rule:
    1. If there is a positive supply side shock $(v_t \uparrow)$, then $(x_t \uparrow)$ and $(L \uparrow)$
    2. So what is the optimal value of ω such that the loss is minimized?
4. It is easy to see what should our steps be in order to determine optimal ω:
    1. Get the L function affected only by ω: get rid off x_t, π_t
    2. Eq. (9) allows you to get rid off of x_t
    3. You can get rid off of π_t as well by using the AS function (next)

Getting inflation as a function of omega

1. Notice that the AS function, if iterated forward, will lead to

$$\begin{aligned} \pi_t &= \beta \cdot E_t \pi_{t+1} + \lambda x_t + v_t \\ &= \sum_{s=0}^{\infty} \beta^s E_t \left(\lambda x_{t+s} + v_{t+s} \right) \end{aligned}$$

2. Insert eq.(9) into the forward iterated AS curve

$$\begin{aligned} \pi_t &= \sum_{s=0}^{\infty} \beta^s E_t [\lambda(-\omega v_{t+s}) + v_{t+s}] \\ &= \sum_{s=0}^{\infty} \beta^s E_t \left[(1-\lambda\omega) v_{t+s} \right] && (10) \\ &= (1-\lambda\omega) \sum_{s=0}^{\infty} \beta^s E_t v_{t+s} && (11) \end{aligned}$$

Getting inflation as a function of omega (cont.)

1. We have just obtained the following eq.

$$\pi_t = (1 - \lambda\omega) \sum_{s=0}^{\infty} \beta^s E_t v_{t+s}$$

2. Taking into account that

$$E_t v_{t+s} = (\rho_v)^s v_t$$

3. Therefore, we get

$$\pi_t = (1 - \lambda\omega) \sum_{s=0}^{\infty} (\beta\rho_v)^s v_t$$

4. Therefore, we can arrive at our objective

$$\pi_t = (1 - \lambda\omega) \frac{v_t}{1 - \beta\rho_v} = \frac{1 - \lambda\omega}{1 - \beta\rho_v} v_t \tag{12}$$

Inserting back into the Loss function

1. We have by now obtained two main results

$$x_t = -\omega \cdot v_t \quad , \quad \pi_t = \frac{1-\lambda\omega}{1-\beta\rho_v} v_t$$

2. Inserting these two back into the Loss function we get L depending on

$$\omega \text{ and } v_t$$

3. Doing just that (see Appendix 3 for details)

$$L = \frac{1}{2} E_t \left[\sum_{s=0}^{\infty} \beta^s \left(a x_{t+s}^2 + \pi_{t+s}^2 \right) \right]$$

$$\ldots$$

$$L = \frac{1}{2} \left[-a\omega^2 + \left(\frac{1-\lambda\omega}{1-\beta\rho_v} \right)^2 \right] E_t \sum_{s=0}^{\infty} \beta^s v_{t+s}^2$$

$a(-\omega)^2$

4. To min L, take FOC with respect to ω, ignoring the shocks $E_t \sum_{s=0}^{\infty} \beta^s v_{t+s}^2$

Minimizing the Loss function

$a(-\omega)^2$

1. We have obtained that

$$L = \frac{1}{2}\left[-a\omega^2 + \left(\frac{1-\lambda\omega}{1-\beta\rho_v}\right)^2\right] E_t \sum_{s=0}^{\infty} \beta^s v_{t+s}^2$$

2. To min L, take FOC with respect to ω, ignoring the shocks $E_t \sum_{s=0}^{\infty} \beta^s v_{t+s}^2$

$\boxed{\text{See Appendix 4 for details}}$

$+$

$$\frac{\partial L}{\partial \omega} = 0 \Rightarrow -a\omega + \left(\frac{1-\lambda\omega}{1-\beta\rho_v}\right)\frac{-\lambda}{1-\beta\rho_v} = 0 \tag{13}$$

3. Solving for ω in eq.(25), we get

$$\omega = \frac{\lambda}{a(1-\beta\rho_v)^2 + \lambda^2} \tag{14}$$

4. Now, the result in eq. (14) can be inserted into eq.(9) to obtain the optimal values for x_t and π_t under commitment (x_t^c, π_t^c)

The optimal levels of output gap and inflation

1. Insert eq. (14) into eq.(9) to obtain the optimal value for x_t under commitment (x_t^c)

$$\begin{aligned} x_t^c &= -\omega \cdot v_t = -\frac{\lambda}{a(1-\beta\rho_v)^2+\lambda^2}v_t \\ x_t^c &= -\lambda\phi^c v_t \end{aligned} \tag{15}$$

with

$$a^c = a(1-\beta\rho_v) \text{ and } \phi^c = \frac{1}{a^c(1-\beta\rho_v)+\lambda^2}$$

2. From eq (12) we get the optimal value for π_t^c

$$\pi_t^c = \frac{1-\lambda\omega}{1-\beta\rho_v}v_t = a^c\phi^c v_t \tag{16}$$

Comparing results: inflation

1. In order to compare results between discretion and commitment, notice that from eq. (15) and (16), we can get

$$x_t^c = -\frac{\lambda}{a^c}\pi_t^c \tag{17}$$

2. Similar to the result in the discretion case, eq.(4).:

$$x_t^d = -\frac{\lambda}{a}\pi_t^d$$

3. A crucial difference between the two cases. As $a^c = a(1-\beta\rho_v)$, and $\beta\rho_v$ are positive parameters, then:

$$a > a^c$$

4. Therefore, for a given level of the output gap, $x_t^c = x_t^d = \bar{x}_t$, the level of π is lower under commitment than under discretion

$$\pi_t^c = \frac{a^c}{\lambda}\bar{x}_t < \pi_t^d = \frac{a}{\lambda}\bar{x}_t \tag{18}$$

Comparing results: two more points

1. For a given supply shock (v_t), the responses of π_t^c, x_t^c are as follows

$$\begin{aligned} x_t^c &= -\lambda\phi^c v_t > x_t^d = -\lambda\phi v_t \\ \pi_t^c &= a^c\phi^c v_t \gtrless \pi_t^d = a\phi v_t \end{aligned}$$

   because $|-\lambda\phi^c| > |-\lambda\phi|$, but $a^c\phi^c \gtrless a\phi$ depending on the values that a and λ assume. **(See figures 3 a 5)**

2. The reaction of the CB against inflation expectations, by increasing interest rates, **is higher than in discretion**

$$r_t^c = \psi^c E_t \pi_{t+1} + \frac{1}{\varphi}\mu_t \qquad (19)$$

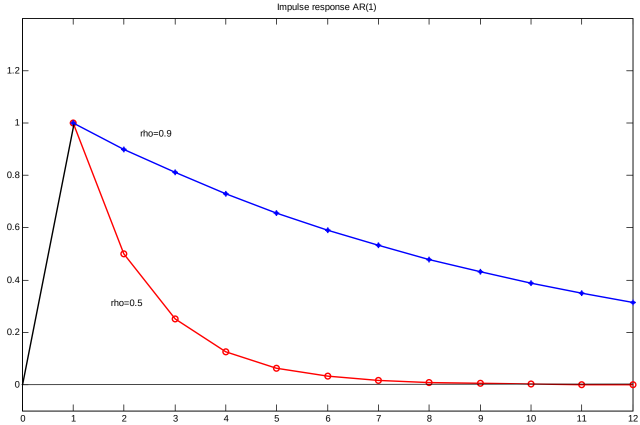
negative

$$\text{because } \psi^c > \psi^d$$

3. Examples of the reaction of (x_t, π_t) to a temporary and unanticipated ~~positive~~ supply shock$(\uparrow v_t)$ in both cases: commitment and discretion. **(See figures next)**

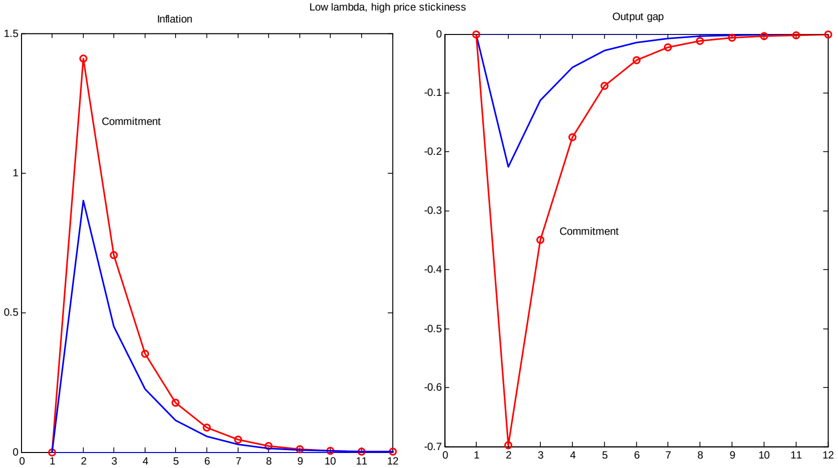
The impact of ~~positive~~ negative supply shock

In fact, the impulse responses of (x_t, π_t) are totally determined by the response of the supply shock (v_t) to a unit shock in $\varepsilon_1 = 1; \varepsilon_{-1} = 0, \varepsilon_{+1} = 0.$



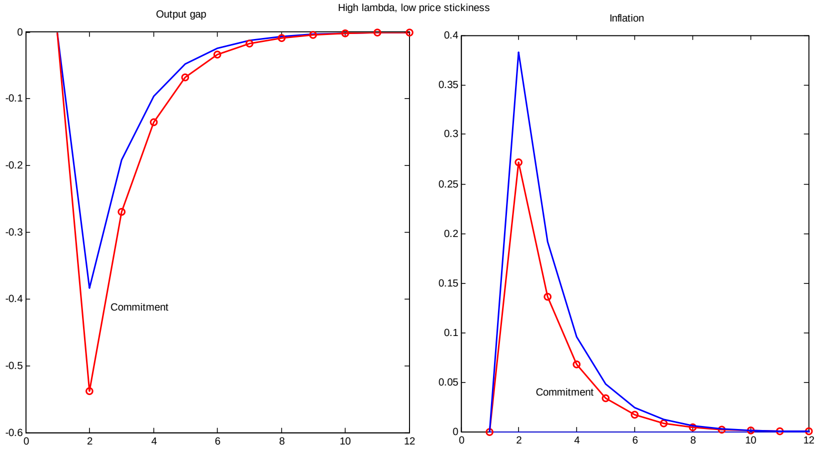
Impulse responses: the case of high price stickiness

Price stickiness: $\lambda = 0.2$; other parameters: $a = 0.8$, $\beta = 0.99$, $\rho_v = 0.5$.



Impulse responses: the case of low price stickiness

Price stickiness: $\lambda = 0.8$; other parameters: $a = 0.8$, $\beta = 0.99$, $\rho_v = 0.5$.



Comparing results: final comments

1. **Compared to discretion**, if the CB commits to the current ad-hoc rule for conducting monetary policy, we get:
    1. lower π for given x (vide eq.18),
    2. tougher responses by the CB to inflation expectations (vide eq.19),
    3. mixed results as far as social welfare is concerned (the impact of volatility upon the Loss function) as highlighted by the previous figures.
2. However, the rule was chosen in an ad-hoc manner. What we want to know is this:
   what is the rule that once chosen, and followed with commitment over time, leads to the highest level of social welfare, that is the lowest possible level for the L function.
3. We call this rule the **"optimal rule"** or the **"best global rule"**.
4. That's what we are going to discuss next.

Appendix 3

1. Iterating forward the AS function, will lead to

$$\begin{aligned}\pi_t &= \beta \cdot E_t \pi_{t+1} + \lambda x_t + v_t \\ &= \sum_{s=0}^{\infty} \beta^s E_t \left(\lambda x_{t+s} + v_{t+s}\right)\end{aligned}$$

2. Inserting eq.(9) into this result we get

$$\begin{aligned}\pi_t &= \sum_{s=0}^{\infty} \beta^s E_t [\lambda(-\omega v_{t+s}) + v_{t+s}] \\ &= \sum_{s=0}^{\infty} \beta^s E_t \left[(1-\lambda\omega) v_{t+s}\right] \qquad (20) \\ &= (1-\lambda\omega) \sum_{s=0}^{\infty} \beta^s E_t v_{t+s} \qquad (21)\end{aligned}$$

Appendix 3 (cont.)

1. Taking into account that

$$E_t v_{t+s} = \left(\rho_v\right)^s v_t$$

2. Therefore, we obtain

$$\pi_t = (1 - \lambda\omega) \sum_{s=0}^{\infty} \left(\beta\rho_v\right)^s v_t$$

3. As $\sum_{s=0}^{\infty} \left(\beta\rho_v\right)^s v_t$ is a geometric series — common ratio $= \beta\rho_v$,initial value $= v_t$ — we get

$$\begin{aligned} \pi_t &= (1 - \lambda\omega) \frac{v_t}{1 - \beta\rho_v} && (22) \\ &= \frac{1 - \lambda\omega}{1 - \beta\rho_v} v_t && (23) \end{aligned}$$

Appendix 4

1. Starting with the L function

$$L = \frac{1}{2} E_t \left[\sum_{s=0}^{\infty} \beta^s \left(a x_{t+s}^2 + \pi_{t+s}^2 \right) \right]$$

2. Knowing that

$a(-\omega)^2$

$$\begin{aligned} x_t &= -\omega \cdot v_t \\ \pi_t &= \frac{1-\lambda\omega}{1-\beta\rho_v} v_t \end{aligned}$$

3. Then

$$\begin{aligned} L &= \frac{1}{2} E_t \left(\sum_{s=0}^{\infty} \beta^s \left\{ a \left(-\omega v_{t+s} \right)^2 + \left[\frac{1-\lambda\omega}{1-\beta\rho_v} v_{t+s} \right]^2 \right\} \right) \\ L &= \frac{1}{2} \left[-a\omega^2 + \left(\frac{1-\lambda\omega}{1-\beta\rho_v} \right)^2 \right] E_t \sum_{s=0}^{\infty} \beta^s v_{t+s}^2 \qquad (24) \end{aligned}$$

Appendix 4

1. To min L, take FOC with respect to ω, ignoring the shocks $E_t \sum_{s=0}^{\infty} \beta^s v_{t+s}^2$

$+$

$$
\begin{aligned}
\frac{\partial L}{\partial \omega} &= 0 && (25) \\
-a\omega + \left(\frac{1-\lambda\omega}{1-\beta\rho_v} \right) \frac{-\lambda}{1-\beta\rho_v} &= 0 && (26)
\end{aligned}
$$

2. Solving for ω in eq.(25), we get

$$
\omega = \frac{\lambda}{a(1-\beta\rho_v)^2 + \lambda^2} \tag{27}
$$

Optimal monetary policy: optimal commitment

The optimal rule problem

1. To obtain the **best global rule**, the central bank has to solve the dynamic optimization problem given by

$$\min \frac{1}{2} E_t \left[\sum_{s=0}^{\infty} \beta^s \left(a x_{t+s}^2 + \pi_{t+s}^2 \right) \right] \qquad \text{(Objective)}$$

subject to

$$\begin{aligned} x_t &= -\varphi \left(r_t - E_t \pi_{t+1} \right) + E_t x_{t+1} + \mu_t & \text{(IS)} \\ \pi_t &= \beta \cdot E_t \pi_{t+1} + \lambda x_t + v_t & \text{(AS)} \end{aligned}$$

2. ... with no other restrictions upon the min problem.
3. In the discretion case, the bank assumed expectations **as given**
4. In the commitment to a simple rule, the rule was stipulated in an **ad-hoc manner**
5. Now, no ad-hoc rules, and expectations are endogenous: the central bank has to obtain an optimal rule and stick to it forever.

The Lagrangian for the optimal rule

1. Set the Lagrangian for this problem

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{2}E_t\left\{\sum_{s=0}^{\infty}\beta^s\left[\left(ax_{t+s}^2+\pi_{t+s}^2\right)\right.\right.\\ & \left.\left.+2q_{t+s}(\pi_{t+s}-\beta E_t\pi_{t+s+1}-\lambda x_{t+s}-v_{t+s})\right]\right\}\end{aligned}$$

   where q_{t+s} is the Lagrangian multiplier associated with the AS curve and is multiplied by 2 in order to compensate for the term $1/2$ outside ...

2. **We do not need the IS function**; it turns out to be irrelevant for this stage of the optimization problem.
3. If we insert the IS into the Lagrangian, take a partial derivative with respect to r_t and make it equal to 0, any value of r_t satisfies this condition
4. The IS is not a binding constraint for this problem; but we need it **at a later stage**

A useful point and the Lagrangian

1. Before we move on, it's better to clarify one point: inflation at $t+s$ is equal to the expected level at $t+s$ plus the forecasting error at $t+s$,

$$\begin{aligned} \pi_{t+s} &= E_t \pi_{t+s} + v_{t+s}, \quad \text{or} \\ E_t \pi_{t+s} &= \pi_{t+s} - v_{t+s} \end{aligned}$$

2. Write the Lagrangian for two consecutive periods

$$\begin{aligned} \mathcal{L} = \; & ... + \beta^0 \{ \left(a x_t^2 + \pi_t^2 \right) + 2 q_t [\pi_t - \beta \left(\pi_{t+1} - v_{t+1} \right) - \lambda x_t - v_t] \} + \\ & \beta^1 \{ \left(a x_{t+1}^2 + \pi_{t+1}^2 \right) + 2 q_{t+1} [\pi_{t+1} - \beta \left(\pi_{t+2} - v_{t+2} \right) - \lambda x_{t+1} - \end{aligned}$$

3. Notice: there are no expectations terms in the $\mathcal{L}$ function

First order conditions (FOC): x

1. As far as x_{t+s} is concerned, for successive periods the FOCs are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_t} &= 2\beta^0 a x_t - 2\beta_t^0 q_t \lambda &&= 2\beta^0 (a x_t - q_t \lambda) = 0 \\
\frac{\partial \mathcal{L}}{\partial x_{t+1}} &= 2\beta a x_{t+1} - 2\beta q_{t+1} \lambda &&= 2\beta^1 (a x_{t+1} - q_{t+1} \lambda) = 0 \\
\frac{\partial \mathcal{L}}{\partial x_{t+2}} &= 2\beta^2 a x_{t+2} - 2\beta_{t+2}^2 q_{t+2} \lambda &&= 2\beta^2 (a x_{t+2} - q_{t+2} \lambda) = 0
\end{aligned}$$

2. As an intertemporal rule, we get the following the result for x_{t+s}

$$a x_{t+s} - q_{t+s} \lambda = 0 \tag{28}$$

First order conditions (FOC): pi

1. Now let´s do the same for the optimal inflation rate

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \pi_t} &= 2\beta^0 \pi_t + 2\beta_t^0 q_t && = 2\beta^0(\pi_t + q_t) = 0 \\
\frac{\partial \mathcal{L}}{\partial \pi_{t+1}} &= 2\beta^1 \pi_{t+1} + 2\beta^1 q_{t+1} - 2\beta^0 \beta q_t && = 2\beta^1(\pi_{t+1} + q_{t+1} - q_t) = 0 \\
\frac{\partial \mathcal{L}}{\partial \pi_{t+2}} &= 2\beta^2 \pi_{t+2} + 2\beta^2 q_{t+2} - 2\beta^1 \beta q_{t+1} && = 2\beta^2(\pi_{t+2} + q_{t+2} - q_{t+1}) = 0
\end{aligned}
$$

1. As a rule, we get the following results for the optimal level of π_{t+s}

$$\pi_{t+s} + q_{t+s} = 0, \quad s = 0 \tag{29}$$

$$\pi_{t+s} + q_{t+s} - q_{t+s-1} = 0, \quad s > 0 \tag{30}$$

2. Notice: **two different rules** depending on what period is considered; one for $s = 0$ (the initial period) and another for all subsequent periods $(s > 0)$

The optimal interest rate rule

1. For the initial period $(s = 0)$, combining eq.(28) with (29), we get (superscript "o" for *optimal*)

$$x_t^o = -\frac{\lambda}{a}\pi_t^o \tag{31}$$

2. ... the **same kind of result** that we met in the cases of discretion and commitment to a simple rule.
3. There is however, a crucial difference between eq.(31) and those for discretion and commitment: eq.(17) and (4)
4. In our current case, the optimal interest rate is **path dependent**: is dependent not only on future expectations but also current conditions
5. Substitute eq.(31) into the AS function and

$$r_t^o = \left[1 - \frac{\lambda}{a\varphi}\right] E_t \pi_{t+1}^o + \frac{1}{a\varphi} x_t^o + \frac{1}{\varphi}\mu_t$$

Path dependence optimal rule

1. The path dependence mechanism can also be in the optimal rule for the second period $(s = 1)$.
2. Combine eq. (28) with (30) and get

$$x_{t+1} - x_t = -\frac{\lambda}{a}\pi_{t+1} \tag{32}$$

3. ... the decisions of the CB at t are determined not only by the conditions prevailing at t, but also by the conditions the bank like to see in the future (x_{t+1}, π_{t+1}).
4. Once the CB sets in its optimal rule for $t + s$, with $s = 1, ..., \infty$, the optimal level of r_t^o can be determined by inserting (32) into the AS

$$r_t^o = \left[1 - \frac{\lambda}{a\varphi}\right] E_t \pi_{t+1}^o + \frac{1}{\varphi}\mu_t \tag{33}$$

Path dependence: major results

1. **Three major implications** from eq (32):
    1. If prices change by one percentage point ($\Delta\pi_{t+1} = 1$), the growth rate of output $(x_{t+1} - x_t)$ should decline by $(-\frac{\lambda}{a})$ percentage points, in order to turn down inflation
    2. Prices remain constant only if output remains constant

    $$\Delta\left(x_{t+1} - x_t\right) = 0 \Rightarrow \Delta\pi_{t+1} = 0$$

    3. **State dependence pricing**: optimal inflation depends upon previous economic conditions, which calls for **policy inertia**: that is the fight against inflation should be gradual over time
2. Let's explain this last result in more detail.

Policy inertia

1. Assume there is a positive technological shock at $t : v_t > 0$
2. From eq. (33), the CB should not react to this type of shocks
3. How does the economy goes back to the equilibrium?
4. The AS tells us that if $v_t > 0$, inflation should increase:

$$\uparrow v_t \Rightarrow \uparrow \pi_t$$

5. But from (32), we know that as x_{t-1} remains constant, then

$$\uparrow \pi_t \Rightarrow \downarrow x_t$$

6. But, from the IS, we know that if $\downarrow x_t$, and as r_t remains constant, then

$$\downarrow E_t \pi_{t+1}, \downarrow E_t x_{t+1}$$

7. Therefore, if no more shocks hit the economy $\downarrow \pi_{t+1}, \downarrow x_{t+1}$, and the process stops only when the equilibrium condition is restored again

The advantages of optimal rules

1. The CB can control inflation and the output gap with **smaller changes in interest** rates, due to the gradual adjustment explained in the previous slide
2. From eq. (33), a one percent increase in inflation expectations, leads to an increase in interest rates that is lower than one percent:

$$\frac{\partial r_t^o}{\partial E_t \pi_{t+1}^o} = 1 - \frac{\lambda}{a\varphi} < 1$$

3. This is in opposition to discretion and simple commitment, where

$$\frac{\partial r_t^c}{\partial E_t \pi_{t+1}^c} > \frac{\partial r_t^d}{\partial E_t \pi_{t+1}^d} > 1$$

4. Therefore, as a final result we get

$$\frac{\partial r_t^o}{\partial E_t \pi_{t+1}^o} < \frac{\partial r_t^d}{\partial E_t \pi_{t+1}^d} < \frac{\partial r_t^c}{\partial E_t \pi_{t+1}^c}$$

Stability and indeterminacy

Untill now

1. Until now, what we have done is to characterize the steady state:
    1. What kind of relationship exists between the endogenous variables in the steady state
    2. How they behave in response to the occurrence of exogenous shocks
2. These are crucial issues ... but there are other problems that should also be answered
3. Remember the three crucial questions ... in dynamic models (whether deterministic or stochastic):
    1. Q1. Does the system under study have a long term equilibrium?
    2. Q2. Is there just one equilibrium, or there is multiple equilibria?
    3. Q3. Is the equilibrium (or equilibria) stable or unstable?

Stability and indeterminacy

1. What we have done so far is to give answers to the first two questions: Yes, there is an equilibrium, and our REM (because they are linear) have just one equilibrium.
2. Nothing was said about the stability and the related issue of indeterminacy.
3. **The study of stability is left as an exercise**
4. Apply the results from the material "Solution to Rational Expectation Models", and take into account the following:
5. *First*, consider only the deterministic part of each model (discretion, commitment, and optimal commitment)
6. *Second*, use only the reduced form solution of each model: the AS and the IS. Notice that you have to insert the expression for the optimal interest rate in each model back into the IS, such that you end up with a system of two equations and two unknowns

Stability and indeterminacy (continued)

1. *Thirdly*, write down the system as we did in the "Solutions to RE Models"

$$\mathbf{E}_t\mathbf{z}_{t+1} = \mathbf{W} \cdot \mathbf{z}_t$$

2. In order to avoid explosive behavior or multiple solutions (indeterminacy) if the system is of dimension 3, for example, vector $\mathbf{z}_t$ contains:
    1. 2 state or predetermined (or backward looking) variables
    2. 1 control (forward looking) variable
3. We need, for a unique and stable solution in this example, to have:
    1. $\mid \lambda_1 \mid > 1,$ for the forward looking
    2. $\mid \lambda_2, \lambda_3 \mid < 1,$ for the backward looking

Stability and indeterminacy (continued)

1. In the three cases studied of the NKM, we have two forward looking variables (π_t, x_t), and no backward looking variables. So for a unique and stable solution we need

$$| \lambda_1, \lambda_2 | > 1$$

2. *Fourth*, calculate the eigenvalues of matrix $\mathbf{W}$ and conclude accordingly.