David Peel, "Advanced Macroeconomics", unpublished manuscript, University of Lancaster, UK

Chapter two

2

Solving Rational Expectations Models

In Chapter 1 we defined the rational expectations hypothesis (REH) as the assumption that people's subjective probability distributions about future outcomes are the same as the actual probability distributions conditional on the information available to them. In practice we will be concerned with moments of these distributions, most frequently the mean, but also occasionally the variance, and very rarely the higher moments (the skewness and so on). When people talk about 'expectations' in popular discourse they mean some single number for the future outcome that is expected to occur; but a moment's thought shows at once that this is not a sensible definition of expectations. Since the future is governed by chance, this exact number will only occur by chance. A better definition would be that 'the expectation' is some summary measure of what may happen, that is, of the probability distribution. Such a summary measure would be, for the central tendency of the distribution, the mean (the first moment); and for the tendency for dispersion around the mean, the variance (the second moment). Then either implicitly or explicitly there would be an indication of the asymmetry of the distribution (its skewness or third moment), and its truncation at the tails (its kurtosis, or the fourth moment). These moments of a distribution over x_t are, respectively, Ex_t, $E(x_t - E\ x_t)^2$, $E(x_t - E\ x_t)^3$, $E(x_t - E\ x_t)^4$ and so on for higher moments. Hence 'expectations' are shorthand for some measure of a probability distribution; in practice we use the mean as the main summary measure, assuming that the other moments are known in some way — this implies that the main feature of the distribution that is changing over time is its mean.

The mathematical term for the mean of a distribution is its 'expected value', and it is usual for applied work on the REH to identify the 'expectation of x_{t+i} (x at time $t + i$)' with the mathematically expected value of x_{t+i}. In this book we shall use the notation $E_{t+j}x_{t+i}$ for expectations framed for the period $t+i$, on the basis of information generally available at time $t+j$; j, i can be positive or negative. E is the mathematical expectations operator, meaning 'mathematically expected value of'. Formally $E_{t+j}x_{t+i}$ is defined as $E(x_{t+i} \mid \Phi_{t+i})$ where Φ_{t+j} is the set of generally available information at time $t+j$. Of course, once x_{t+i} is part of the information set Φ_{t+j}, then $E_{t+j}x_{t+i} = x_{t+i}$ trivially.

If we wish to indicate that the information available to those framing expectations is restricted to a set θ_{t+j} at $t+j$, we shall write $E_{t+j}(x_{t+i} \mid \theta_{t+j})$, that is, the expectation of x at $t+i$ framed on the basis of information set θ available at $t+j$. It is natural to think of $E_{t+j}x_{t+i}$ as 'expectations formed at $t+j$ of x at $t+i$'; this will do for some purposes but it is not quite accurate. It is not in fact the date at which expectations are formed that matters but rather the date of the information set on the basis of which they are formed. Because of information lags, people may form expectations for this period on the basis of last period's information, and we would write this as $E_{t-1}x_t$.

Suppose for extreme simplicity that the model of x_t is:

$$x_{t+1} = x_t + \varepsilon_{t+1} \tag{1}$$

where ϵ_t is normally distributed with a mean of 0, a constant variance of σ^2, independence between successive values, and independence of all previous events; that is, $\varepsilon_t : N(0, \ \sigma^2)$, $E(\varepsilon_t + \varepsilon_{t+j}) = 0$ $(i \neq j)$ and $E\varepsilon_{t+i} \mid \Phi_{t+j} = 0$ $(i > j)$. Equation (1) states that x_t follows a 'random walk' (the change in x_t is random).

The expectation of x_t at t is x_t if we assume that people know x_t then. They cannot know ε_{t+1} because it has not yet occurred and as a random variable its expected value is zero. If we write Φ_t as the total information set at t, then:

$$E_t x_{t+1} = E(x_{t+1} \mid \Phi_t) = (x_t + \varepsilon_{t+1} \mid \Phi_t) = x_t + E(\varepsilon_{t+1} \mid \Phi_t) = x_t \tag{2}$$

$E_t x_{t+1}$ will be an unbiased predictor of x_{t+1}, that is, the mean (or expected value) of the prediction error $x_{t+1} - E_t x_{t+1}$ is zero. Thus:

$$E(x_{t+1} - E_t x_{t+1}) = E(x_t + \varepsilon_{t+1} - x_t) = E\varepsilon_{t+1} = 0 \tag{3}$$

$E_t x_{t+1}$ will also be the efficient predictor of x_{t+1}, that is, the variance of the predictor error is smaller than that of any other predictor. Thus:

$$\text{Variance } (x_{t+1} - E_t x_{t+1}) = E(x_{t+1} - E_t x_{t+1})^2 = E(\varepsilon_{t+1})^2 = \sigma^2 \tag{4}$$

This is the minimum variance possible in prediction of x_{t+1} because ε_{t+1} is distributed independently of previous events (the meaning of 'unpredictable'). Suppose we add any expression whatsoever, say βz_t, where z_t is a variable taken from Φ_t, to $E_t x_{t+1}$, making another predictor $\widehat{x}_{t+1}$:

$$\widehat{x}_{t+1} = x_t + \beta z_t \tag{5}$$

Then:

$$E(x_{t+1} - \widehat{x}_{t+1})^2 = E(\varepsilon_{t+1} - \beta z_t)^2 = \sigma^2 + \beta^2 E z_t^2 \tag{6}$$

The variance will be increased by the variance of the added expression, because this must be independent of ε_{t+1}.

The unbiasedness and efficiency of their forecasts are the two key properties of rational expectations forecasts that we will constantly return to in this book. However for the time being, in this chapter, we shall restrict ourselves to explaining how the rational expectation of variables determined in more complex models is to be found, and how those models are accordingly to be solved.

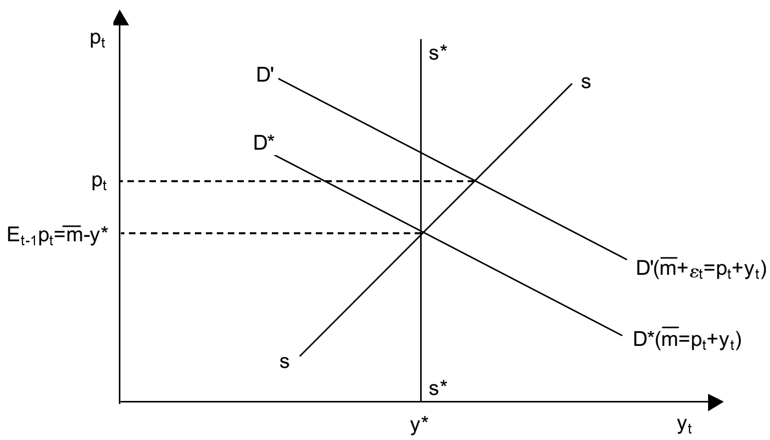


Figure 2.1: A simple macroeconomic model illustrated

THE BASIC METHOD

Now take a simple macro model (illustrated in Figure 2.1):

$$m_t = p_t + y_t \tag{7}$$

$$p_t = E_{t-1} p_t + \delta(y_t - y^*) \tag{8}$$

$$m_t = \overline{m} + \epsilon_t \tag{9}$$

where $m,\ p,\ y$ are the logarithms of money supply, the price level, and output respectively; y^* is normal output, $\overline{m}$ is the monetary target (both are assumed to be known constants). Equation (7) is a simple money demand function with a zero interest elasticity and a unit income elasticity: in Figure 2.1 it is drawn as an aggregate demand curve with a slope of -1. Equation (9) is a money supply function in which the government aims for a monetary target with an error, ε_t, which has the properties of our previous ε in (2.1). As the error shocks the economy, aggregate demand shifts up and down around D^*D^*, its steady-state position set by $\overline{m}$. Equation (8) is a Phillips curve as can be seen by subtracting p_{t-1} from both sides; in this case it states that the rate of inflation equals last period's expectation of the inflation rate plus a function of 'excess demand'. We can think of the 'periods' as being 'quarters' and prices as being set, as quantities change, on the basis of last quarter's information about the general price level — hence we appeal to an information lag of one quarter and E_{t-1}refers to this quarter's expectation formed (the operative element) on the basis of last quarter's information. In Figure 2.1, equation (8) is drawn as the aggregate supply curve; rising output requires rising prices, given expected prices, because each producer wants his own relative price to be higher to compensate for the extra effort of higher supply. The vertical supply curve, S^*S^*, is the long-run Phillips curve, indicating that when producers know what the general price level is they will not be 'fooled' into supplying more output as it rises because they realize their own relative price is unchanged.

This model has three linear equations with three endogenous variables, two exogenous variables, $\overline{m}$ and ε_t, and an expectation variable, $E_{t-1}p_t$. Given the expectation, we can solve it normally, for example, by substitution. So substituting for $m_t$and p_t from (8) and (9) into (7) gives us:

$$\overline{m} + \varepsilon_t = E_{t-1}p_t + (1+\delta)y_t - \delta y^* \tag{10}$$

This corresponds to the intersection of the $D'D'$ and SS curves in Figure 2.1. But we now need to find $\mathrm{E}_{t-1}\mathrm{p}_t$, to get the full solution.

To do this, we write the model in expected form (i.e. taking expectations at $t-1$ throughout) as:

$$E_{t-1}m_t = E_{t-1}p_t + E_{t-1}y_t \tag{7^e}$$

$$E_{t-1}p_t = E_{t-1}p_t + \delta(E_{t-1}y_t - y^*) \tag{8^e}$$

$$E_{t-1}m_t = \overline{m} \tag{9^e}$$

Substituting $(8)^e$ and $(9)^e$into $(7)^e$ gives:

$$E_{t-1}p_t = \overline{m} - y^* \tag{11}$$

This is the intersection of the D^*D^* and S^*S^* curves in Figure 2.1. S^*S^* shows what producers will supply on the assumption that the prices they receive are those which they expect (this is an 'expected supply' curve); D^*D^* shows what output will be demanded at different prices on the assumption that the money supply is $\overline{m}$, as expected — (an 'expected demand' curve). Where these two curves intersect is accordingly expected output and prices.

Equation (11) is substituted into (10) to give the full meaning of the p_t, y_t intersection in figure 2.1:

$$y_t = y^* + \frac{1}{1+\delta}\varepsilon_t \tag{12}$$

Consequently, from (8) and using (11):

$$p_t = \overline{m} - y^* + \frac{\delta}{1+\delta}\varepsilon_t \tag{13}$$

The solutions for y_t and p_t consist of an expected part (y^* and $\overline{m} - y^*$, respectively) and an unexpected part (functions of ε_t). Rational expectations has incorporated anything known at $t-1$ with implications for p and y at time t into the expected part, so that the unexpected part is purely unpredictable.

This model, though simple, has an interesting implication, first pointed out by Sargent and Wallace (1975). The solution for y_t is invariant to the parameters of the money supply rule. Output would in this model be at its normal level in the absence of surprises, which here are restricted to monetary surprises. If the government attempts to stabilise output by changing the money supply rule to, say,

$$m_t = \overline{m} - \beta(y_{t-1} - y^*) + \varepsilon_t \tag{14}$$

then still the solution for output is (12), because this money supply rule is incorporated into people's expectations at $t-1$ and cannot cause any surprises. The only effect is on expected (and so also actual) prices:

$$E_{t-1}p_t = \overline{m} - \beta(y_{t-1} - y^*) - y^* = \overline{m} - y^* - \frac{\beta}{1+\delta}\varepsilon_{t-1} \tag{15}$$

$$p_t = \overline{m} - y^* - \frac{\beta}{1+\delta}\varepsilon_{t-1} + \frac{\delta}{1+\delta}\varepsilon_t \tag{16}$$

Note that this will raise the variance of prices around their long-run value $\overline{m} - y^*$ by $(\frac{\beta}{1+\delta})^2\sigma^2$. This is illustrated in Figure 2.2, where we start this

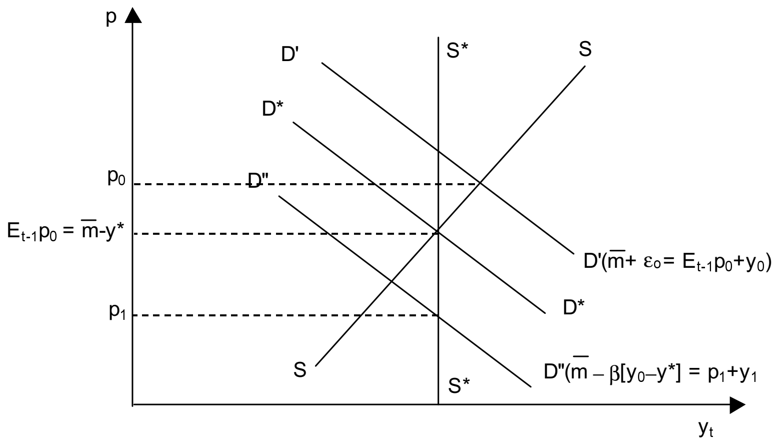


Figure 2.2: A simple model with an interventionist monetary rule

model out from steady state at period $t-1$, where $E_{t-1}p_0 = \overline{m} - y^*$, $E_{t-1}y_0 = y^*$; let there be a shock in period 0, ε_0.

The solution for $(p_0,\ y_0)$ is the same as before. But now the government responds in period 1 with a money supply contraction, reducing m to $\overline{m} - \frac{\beta}{1+\delta}\varepsilon_0$; this shifts the aggregate demand curve to $D''D''$. But because producers know in period 0 that this reaction will occur, they work out the intersection of their expected supply S^*S^*, and the expected demand, $D''D''$, correctly anticipating that $p_1 = \overline{m} - \frac{\beta}{1+\delta}\varepsilon_0 - y^*$. Where these curves intersect is accordingly expected output and prices.

This of course contradicts the well-known results for models with backward-looking expectations whereby stabilization policy by government can reduce fluctuations in output, provided the government chooses the appropriate monetary target. For example, suppose we had assumed in accordance with the popular practice of the 1960s that expectations of the price level were formed adaptively. The adaptive expectations hypothesis is that:

$$x_t^e - x_{t-1}^e = \mu(x_{t-1} - x_{t-1}^e) \qquad 0 < \mu < 1 \tag{17}$$

or that expectations of x_t change by some positive fraction, μ, of last period's error. This can be written equivalently as:

$$\begin{aligned} x_t^e &= \mu x_{t-1} + (1-\mu)x_{t-1}^e = \mu x_{t-1} + (1-\mu)[\mu x_{t-2} + (1-\mu)x_{t-2}^e] \\ &= \mu \sum_{i=0}^{\infty} (1-\mu)^i x_{t-1-i} \end{aligned} \tag{18}$$

by continuous substitution for $x^e_{t-2}, x^e_{t-3} \dots$

Substituting p^e_t for $E_{t-1}p_t$ in our simple model of (7) to (9) turns it into an orthodox dynamic model to be solved by standard methods. Equation (8) becomes:

$$p_t = \mu \sum_{i=0}^{\infty} (1-\mu)^i p_{t-1-i} + \delta(y_t - y^*) \tag{2.8^a}$$

We can see that expected prices depend not on the planned money supply but on past events (past prices), which were known to the government last period. Consequently the government can plan a money supply for this period confident that it will not be 'frustrated' by a response from expectations. They can set a target m^*, such that $y_t = y^*$. This will be a target which accommodates prices at their expected level, delivering $p_t = p^e_t$; for $(8)^a$ assures us that when $p_t = p^e_t$, $y_t = y^*$. By (7), when $p_t = p^e_t$ and $y_t = y^*$, then:

$$m^* = p^e_t + y^* = \mu \sum_{i=0}^{\infty} (1-\mu)^i p_{t-1-i} + y^* \tag{19}$$

We now find that the solution for output depends on the deviations of money supply from this optimal target:

$$y_t = y^* + \frac{1}{1+\delta}(m_t - m^*) \tag{20}$$

These deviations may be due either to unpredictable errors, ε, as in the rational expectations (RE) case, or to a policy failure to plan m_t at m^*; in other words:

$$m_t - m^* = \varepsilon_t + m^T - m^* \tag{21}$$

where m^T is the actual policy target. But in this adaptive model both affect output, whereas in the RE version only ε, the error term, does. In other words, the monetary policy chosen affects output not, as in the RE case, merely the monetary surprise.

In subsequent chapters we shall be examining this RE model and a number of considerably more complex RE models whose properties will differ from this one substantially. Nevertheless it is a common feature of all these models that there is an important difference between the effects of an anticipated and of an unanticipated change in any exogenous variable; by contrast, in models where expectations are formed adaptively (or as any fixed function of past data) it makes no difference. This is probably the most fundamental result of rational expectations. It is the nature of the difference of these effects that forms the detailed study of RE models.

The method of solution set out above (the 'basic' method) will suffice for all RE models in which there are expectations (at any date in the past) of current events only. To repeat, this method involves three steps:

1. Solve the model, treating expectations as exogenous.
2. Take the expected value of this solution at the date of the expectations, and solve for the expectations.
3. Substitute the expectations solutions into the solution in 1, and obtain the complete solution.

RE MODELS WITH EXPECTATIONS OF FUTURE VARIABLES (REFV MODELS)

It will very often, in fact almost invariably, be the case — in the nature of economic decisions which, as we have seen, involve a view of the future — that expectations of future events, whether formed currently or in the past, will enter the model. For these REFV models, our basic method must be supplemented and it can be replaced by more convenient alternatives.

For example, add to our previous simple model the assumption made by Cagan (1956) in his influential study of hyperinflation, that the demand for money responds negatively to expected inflation (we can think of this as approximating the effect of interest rates on money demand in less virulent inflations). Let the model now be:

$$m_t = p_t + y_t - \alpha(E_{t-1}p_{t+1} - E_{t-1}p_t)\ (\alpha > 0) \tag{22}$$

$$p_t = E_{t-1}p_t + \delta(y_t - y^*) \tag{8}$$

$$m_t = \overline{m} + \varepsilon_t \tag{9}$$

We keep (8) and (9) as before. In (22) expectations of inflation in the current period are regarded as formed on the basis of last period's (quarter's) information; as in (8) we are appealing to an information lag.

Let us use our basic method and see how it has to be adapted for this model. Step 1 (solving given expectations as exogenous) gives us:

$$\overline{m} + \varepsilon_t = p_t + \frac{1}{\delta}(p_t - E_{t-1}p_t) + y^* - \alpha(E_{t-1}p_{t+1} - E_{t-1}p_t) \tag{23}$$

This is the same intersection ($p_t,\ y_t$) as in Figure 2.1, except for the extra term $-\alpha(E_{t-1}p_{t+1} - E_{t-1}p_t)$ which shifts $D^{'}D^{'}$ relative to what is shown there.

To find $E_{t-1}p_t$ and $E_{t-1}p_{t+1}$ we now take expectations of the model at $t-1$ (step 2) to yield:

$$\overline{m} - y^* = (1+\alpha)E_{t-1}p_t - \alpha E_{t-1}p_{t+1} \tag{24}$$

Equation (24) can solve for $E_{t-1}\ p_t$ in terms of $\overline{m}$, y^*, and $E_{t-1}p_{t+1}$. But this is not a solution because $E_{t-1}p_{t+1}$ is not solved out; we appear to have shifted the problem into the future.

To solve for $E_{t-1}p_{t+1}$ we may lead the model by one period (for example, write (22) as $m_{t+1} = p_{t+1} + y_{t+1} - \alpha(E_t p_{t+2} - E_t p_{t+1})$) and take expectations of it at $t-1$ as before. This yields analogously:

$$\overline{m} - y^* = (1+\alpha)E_{t-1}p_{t+1} - \alpha E_{t-1}p_{t+2} \tag{25}$$

We have now solved for $E_{t-1}p_{t+1}$ in terms of m, y^*, and $E_{t-1}p_{t+2}$ again shifting the problem into the future. This naturally leads us to solving for expected values using the method of forward iteration proposed by Thomas Sargent. We write (25) as:

$$E_{t-1}p_{t+1} = \frac{1}{1+\alpha}(\overline{m} - y^*) + \frac{\alpha}{1+\alpha}E_{t-1}p_{t+2} \tag{26}$$

Substitute successively (forwards) for $E_{t-1}p_{t+2}$, $E_{t-1}p_{t+3}$ and so on in (26) to obtain:

$$E_{t-1}p_t = \frac{1}{1+\alpha}\sum_{i=0}^{N-1}\left(\frac{\alpha}{1+\alpha}\right)^i(\overline{m} - y^*) + \left(\frac{\alpha}{1+\alpha}\right)^N E_{t-1}p_{t+N} \tag{27}$$

Let $N \to \infty$ and assume (as seems natural) that $E_{t-1}p_{t+i}$ is stable, so that $E_{t-1}p_{t+N} \to$ its equilibrium as $N \to \infty$. Since $\left(\frac{\alpha}{1+\alpha}\right)^N \to 0$ also as $N \to \infty$ the final, remainder, term in (27) disappears and (27) becomes:

$$E_{t-1}p_t = \frac{1}{1+\alpha}\sum_{i=0}^{\infty}(\frac{\alpha}{1+\alpha})^i(\overline{m} - y^*) = \overline{m} - y^* \tag{28}$$

We can reach the same result by using the forward operator, B^{-1} (B is the backward operator that instructs us to lag the variable but not the expectations date, unlike L which instructs us to lag both).

Write (24) as:

$$(1+\alpha)\left(1 - \frac{\alpha}{1+\alpha}B^{-1}\right)E_{t-1}p_t = \overline{m} - y^* \tag{29}$$

It follows that:

$$E_{t-1}p_t = \frac{1}{1+\alpha}\frac{\overline{m}-y^*}{1-\frac{\alpha}{1+\alpha}B^{-1}} = \frac{1}{1+\alpha}\sum_{i=0}^{\infty}\left(\frac{\alpha}{1+\alpha}B^{-1}\right)^i(\overline{m}-y^*)$$
$$= \overline{m}-y^* \quad (30)$$

In this particular case, the exogenous variables are constant. However, Sargent's method can be generalised; for example suppose that the money supplies were exogenously given to us (it might be that each period the central bank's policies are reassessed in the light of a variety of current information including bank announcements and the result is most simply written down as a new set of projections each time.) The model is the same, (22) and (8), except for the omission of (9).

(23) now becomes:

$$m_t = p_t + \frac{1}{\delta}(p_t - E_{t-1}p_t) + y^* - \alpha(E_{t-1}p_{t+1} - E_{t-1}p_t) \quad (31)$$

Subtracting from this its expected value at $t-1$ yields:

$$p_t - E_{t-1}p_t = \frac{\delta}{1+\delta}(m_t - E_{t-1}m_t) \quad (32)$$

which tells us that the solution for prices depends on the revision to the money supply planned at t plus expected prices.

To find expected prices take expectations of (31):

$$E_{t-1}m_t - y^* = (1+\alpha)(E_{t-1}p_t) - \alpha(E_{t-1}p_{t+1}) \quad (33)$$

from which we obtain using the forward operator:

$$E_{t-1}p_t = [\frac{1}{1+\alpha}\sum_{i=0}^{\infty}(\frac{\alpha}{1+\alpha})^i E_{t-1}m_{t+i}] - y^* \quad (34)$$

In other words the whole path of future monetary policy foreseen at $t-1$ determines expected prices for t.

We have now seen how in a rational expectations model the expected future affects expectations of the present. Plainly the direction of causation is from the (expected) future to the (expected) present. We note that (24) is a difference equation which had to be 'solved forwards' by iteration into the future; the present depends on the future via a stable root, $\frac{\alpha}{1+\alpha}$. However, it is possible — though on reflection odd — to look at the relationship differently, as one where the expected present affects the expected future. Looking at it this way draws attention to the possibility that a rational expectations model may have self-generating explosive paths or 'bubbles'.

Suppose we go back to (24) and (25). We could have carried on in this way indefinitely and it is easy to see that we would have obtained a series of equations which could be written as a sort of difference equation:

$$\overline{m} - y^* = (1+\alpha)E_{t-1}p_{t+i} - \alpha E_{t-1}p_{t+i+1} \; (i \geqslant 0) \quad (35)$$

This is actually a difference equation in a variable p^e_{t+1}, defined to be p_{t+i} as expected from $t-1$:

$$p^e_{t+i+1} - \frac{1-\alpha}{\alpha} p^e_{t+i} = -\frac{1}{\alpha}(\overline{m} - y^*) \ (i \geqslant 0) \tag{36}$$

The solution of this first-order non-homogenous difference equation is familiarly:

$$p^e_{t+i} = \overline{m} - y^* + [p^e_t - (\overline{m} - y^*)](\frac{1+\alpha}{\alpha})^i \ (i \geqslant 0) \tag{37}$$

where $\overline{m} - y^*$ is the equilibrium of p_t (the 'particular' solution), $\frac{1+\alpha}{\alpha}$ is the unstable root (note that it is the inverse of the stable forward root when we solved the model forwards) and $p^e_t - (\overline{m} - y^*)$ is the constant (determined by the initial value p^e_t) in the 'general' solution. Here we are solving the model 'backwards' from the future to the present in the sense that the (expected) future is depending on the (expected) present; the same root that was stable when the model was solved forwards is now unstable when the model is solved backwards.

This can be understood from Figure 2.3. Here we have drawn the long-run Phillips curve, S^*S^*, and the aggregate demand curve, D^*D^*, on the assumption that prices are not expected to change ($E_{t-1}\Delta p_{t+1} = 0$). On this assumption the expected price level is $\overline{m} - y^*$ as before (see Figure 2.1). But we can rewrite the expected solution of equations (22) and (9) as $E_{t-1}\Delta p_{t+1} = \frac{1}{\alpha}(E_{t-1}p_t + y^* - \overline{m})$ which shows that if expected aggregate demand ($E_{t-1}p_t + y^*$) exceeds the expected money supply, $\overline{m}$, then it must be because prices are expected to rise; and vice versa. So to the right of D^*D^*, prices are expected to rise, and to its left they are expected to fall, as shown by the arrows on Figure 2.3. Since output is always expected to be y^* on S^*S^*, the possible solution for $E_{t-1}p_t$ and subsequent $E_{t-1}p_{t+1}$ are shown by the arrows on S^*S^*. One such solution is shown by the intersection of the $D^{'}D^{'}$ dashed curve showing the expected aggregate demand curve, when prices are expected to rise.

CHOOSING A UNIQUE DEMAND PATH

Equation (37) and Figure 2.3 give an infinite number of solution paths for p^e_{t+i} ($i \geqslant 0$). For we are free to choose any value of p^e_t we like; the model does not restrict our choice. Another way of looking at (37) is to say that we can choose any future value for any p^e_{t+i} we wish and work back from that to a solution for p^e_t. We could already have guessed that this would be so from (24) for, to obtain the expectation of a current value, we were compelled to take a view about $E_{t-1}p_{t+1}$. Any view of

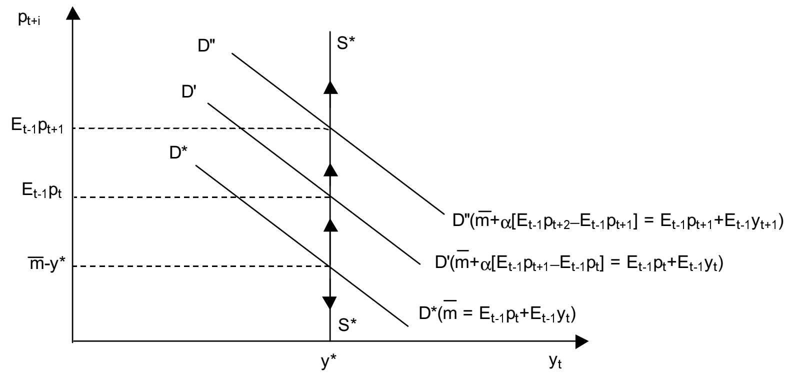


Figure 2.3: The solution expected at $t-1$ of a simple REFV model, illustrated for an unstable path

this future will then compel a present which is consistent with it; any set of expectations is therefore self-justifying.

REFV models (that is, the vast majority) would be little better than curiosa if they did not carry with them additional restrictions sufficient to define a unique solution; for they would merely assert in effect that 'anything can happen provided it is expected, but what is expected is arbitrary'. Worse still, as (37) illustrates, these paths for events can be unstable; in fact, our model here implied that all paths for prices except that for which $p_t^e = \overline{m} - y^*$, explode monotonically as shown in Figure 2.4. Thus our particular REFV model would assert that only by accident would an equilibrium price level be established, otherwise prices would be propelled into either ever-deepening hyperdeflation or ever-accelerating hyperinflation, even though money supply is held rigid! (Output in this model is always expected to be equilibrium.) While such an assertion may appeal to some it has not impressed those who have espoused RE models; they have looked instead for additional restrictions.

We have already hinted at the source of an additional restriction in our model by noting the instability of all but one path. It is clear that the unstable paths are in some sense absurd. The question is: what would prevent them? It has to be the case that behaviour would alter in such a way as to prevent them.

Consider, for example, the path of ever-accelerating hyperinflation anticipated fully now (on the basis of last period's information). People deciding how much money to hold for transactions would expect now that in so many years they will need truckfuls of money to buy the daily groceries; they would therefore find an alternative means of carrying out transactions to avoid the investment in trucks they will otherwise anticipate. They would use beans or cows or sophisticated forms of barter to replace the old money. Ultimately the old money would not be used at all; prices would be defined in the new money, say beans.

But money has an issuer; it may be a bank or a government. The issuer derives profits from people's use of its money issue, and it will pay them to avoid its replacement. This it can only do by stopping any such hyperinflation 'bubbles' occurring. It turns out that a commitment on the issuer's part to put an end to any such inflation at some point, by decreasing the money supply at a sufficient rate to offset any decline in real money balances held, will do the trick. For if people expect that inflation will stop at some period $t+N$ (at which the bank will 'step in'), then this implies an arresting of the very ongoing process that sustains the earlier path. Real money balances desired in $t+N$ will now be higher than anticipated in that path, so inflation must be lower in $t+N$. But if this is so, then real money balances in $t+N-1$ will be similarly higher, so also inflation will be lower then; and so on. The whole path will be invalidated.

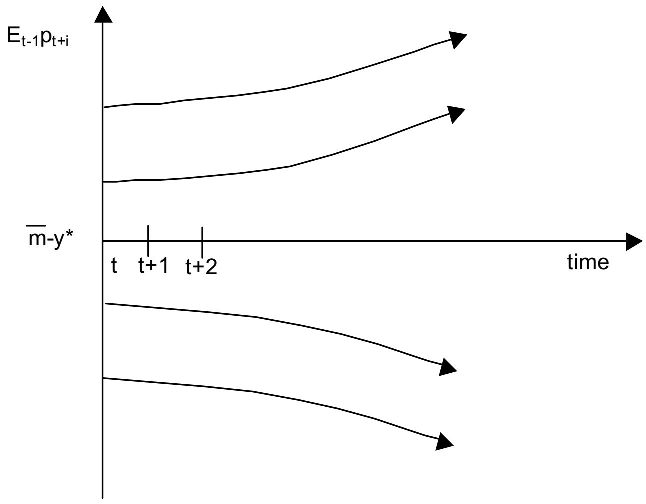


Figure 2.4: The solution paths for the price level expected at $t-1$, as in equation (37)

In fact we can show this formally by imposing on the difference equation (37) the condition that :

$$p^e_{t+i+1} - p^e_{t+i} = 0 \ (i \geqslant N) \tag{38}$$

and letting (37) run from $N \leqslant i \geqslant 0$, since $t + N$ is the period when the bank's new regime takes over. Using (37) for $i = N$, we have:

$$\overline{m} - y^* = p^e_{t+N} + \alpha(p^e_{t+N} - p^e_{t+N+1}) = \text{by (38) } p^e_{t+N} \tag{39}$$

By (37) this implies:

$$\overline{m} - y^* = \overline{m} - y^* + [p^e_t - (\overline{m} - y^*)]\left(\frac{1-\alpha}{\alpha}\right)^N \tag{40}$$

or:

$$p^e_t = (\overline{m} - y^*) \tag{41}$$

It can be seen that (41) when applied to (37) selects the unique stable path for p^e_{t+i} so that:

$$p^e_{t+1} = \overline{m} - y^* \ (i \geqslant 0) \tag{42}$$

An analogous argument can be constructed for the path of ever-deepening hyperdeflation. In this case people will 'demand' infinite amounts of money because its return is infinite in the long term; this implies that the money will be hoarded and disappear from circulation. The bank or government will wish to prevent this (because otherwise some other money will come into existence) by issuing money until the profit rate on the issue has returned to a normal level, that is the rate of deflation is zero. The knowledge that the issuer will go on issuing money until this occurs acts to impose the same condition (38) on the model.

We have constructed verbal arguments to justify the imposition of a 'terminal condition' such as (38) in our model. These arguments appeal to forces not explicitly in the model, but which would be brought into play by certain types of behaviour apparently allowed for by the model. These forces will differ from model to model; for example we may appeal to legal controls or supervisory agencies to ensure 'orderly markets', or to competitive forces[1], or to precepts upon government itself. But an RE model with expectations of the future (REFV model) is incomplete

[1] For example, in the competitive equilibrium model of the labour market of Lucas and Sargent, as set out in, for example, Sargent (1979a, chapter 16) the transversality conditions of households and firms supply the necessary terminal conditions. These conditions are necessary for optimality; in other words, explosive paths for labour supply and demand are not followed by households or firms because they are suboptimal.

without some forces of this kind to supply an additional restriction, such as the terminal condition here.

Another way of describing our 'terminal' condition would be as a 'side' or 'transversality' condition: all these express the same idea, that there is an additional restriction on the model, here coming from government or central bank behaviour designed to rule out what is from their (or society's) viewpoint an undesirable outcome, in this instance for the monetary environment. We will come across other such transversality conditions later in this book (for example, on private or government borrowing designed to rule out unsustainable and thus undesirable borrowing paths).

Our terminal condition (38) has the effect in the model here of selecting the unique stable path. For REFV models with such a unique stable path like the one here (that is, with the 'saddlepath' property, so called because any deviation from this path is unstable), the imposition of terminal stationarity on the expectations ensures the selection of this path. For such models, it is therefore only necessary to specify as a side condition on the model that the solutions be stable or stationary; this condition is referred to variously in the literature as the 'stability' or 'stationarity' or 'convergence' condition, or 'ruling out speculative bubbles' or 'boundedness'. We appealed to it when we used the forward iteration or operator method above; to obtain that solution we had to assume that the expected price sequence was stable.

We have now completed step (2) in our solution procedure, albeit in a more complex manner than before; call it step $(2')$. We proceed to step (3) and substitute for $E_{t-1}p_t$, $E_{t-1}p_{t+1}$ into (23). It turns out in this model that the solution is the same as for our earlier model, as the reader can easily verify.

We may now review our basic method for solving REFV models:

1. Solve the model, treating expectations as exogenous.

2. Take the expected value of this solution at the date of the expectations. If the model generates a unique stable path for the expectational variables, impose the stability condition, and derive this solution for the expectations. Do so either by the forward iteration or the backward difference equation method.

3. Substitute the expectations solutions into the solution in 1 and obtain the complete solution.

BUBBLES

The significance of the terminal condition that enforces stability can be seen by considering 'bubbles' or 'will o' the wisp' variables. Suppose we had no such terminal condition. Then it can be seen that it can be rational to expect at $t-1$ any price level for t provided one also expects an ever-exploding price level for the future, the 'bubble'.

It is possible to add arbitrary variables to the solution of REFV models provided they obey certain processes dictated by the coefficients in the model's future expectations (see, for example, Canzoneri, 1983; and Gourieroux et al., 1982).

For example, take the model we have been looking at, of (22), (8) and (9). Suppose people believe at $t-1$, for no good reason, that prices would be affected by $(\frac{1+\alpha}{\alpha})^i E_{t-1} z_{t+i}$ where:

$$E_{t-1} z_{t+i} = z_{t-1} \tag{43}$$

(that is, z_t is a martingale). Their belief, though 'irrational', would formally be validated by the model, since

$$E_{t-1} p_{t+i} = (\overline{m} - y^*) + (\frac{1+\alpha}{\alpha})^i E_{t-1} z_{t+i} \tag{44}$$

is a solution to the model, as can be verified by substituting (44) and (43) into (24). Any 'will o' the wisp' variable, z_{t-1}, could therefore produce an irrational solution to an REFV model by this self-validating process.

This is simply an implication of the indeterminacy of p_t^e we noted earlier in commenting on equation (37); so we can write $p_t^e - (\overline{m} - y^*) = \mu_{t-1}$ where μ_{t-1} is anything. However, the solution to the bubble problem is one and the same as that of the indeterminacy and instability problem: we have to impose an additional restriction on the model to ensure determinacy and stability. Since the exploding bubble must at some point violate the terminal condition, the whole path collapses back to the unique stable solution.

This terminal condition approach to ruling out bubbles is similar in effect to McCallum's suggestion (1983) that a 'minimum set of state variables' (MSV) be imposed on the solution; that is, one eliminates as many state variables as possible from the solution while still maintaining consistency with the rational expectations constraint. In effect the minimum set excludes any such extraneous variables that enter as bubbles. We would argue that the justification of imposing MSV lies in the optimizing transversality conditions on the agents in the model. However, of course in practice the procedures deliver the same solution.

OTHER METHODS OF SOLUTION FOR REFV MODELS

Not surprisingly there are several other methods for finding the unique stable solution to an REFV model which has one. We shall explain two in detail because they have been widely used: the Muth method of undetermined coefficients and the Lucas method of undetermined coefficients.

The Muth Method of Undetermined Coefficients

The Muth method starts from the proposition that the general solution of our model can be written (via the Wold decomposition — see the time-series annex at the end of the book):

$$p_t = \overline{p} + \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i} \tag{45}$$

$$y_t = \overline{y} + \sum_{i=0}^{\infty} \phi_i \varepsilon_{t-i} \tag{46}$$

where $\overline{p}$, $\overline{y}$ are the equilibrium values of p_t, y_t. Note that this way of writing the solution assumes that there are no expected future exogenous variables or else that they can be entirely substituted out in terms of current and past events. This implies that the Muth and Lucas methods are not entirely general (in particular it means that the forward root is solved backwards and so appears in its inverse, unstable, form), but they are useful for the wide class of models where this assumption is valid.

Let us focus on the solution for p_t, since that for y_t follows easily enough. $\overline{y} = y^*$ and $\overline{p} = \overline{m} - y^*$ by setting $E_{t-1}p_t = E_{t-1}p_{t+1} = p_t = \overline{p}$ and $y_t = \overline{y}$ in the model.

Having found the equilibrium in terms of the constants, we now drop these from the model and define (p_t, y_t) in deviations from equilibrium. The model can now be written in terms of p_t as:

$$\varepsilon_t = (1 + \frac{1}{\delta})p_t + (\alpha - \frac{1}{\delta})E_{t-1}p_t - \alpha E_{t-1}p_{t+1} \tag{47}$$

Using (45):

$$p_t = \sum_{i=0}^{\infty} \pi_t \varepsilon_{t-i} \tag{48}$$

$$E_{t-1}p_t = \sum_{i=1}^{\infty} \pi_i \varepsilon_{t-i} \tag{49}$$

$$p_{t+1} = \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i-1} = \sum_{i=1}^{\infty} \pi_{i+1} \varepsilon_{t-i} \tag{50}$$

$$E_{t-1} p_{t+1} = \sum_{i=1}^{\infty} \pi_{i+1} \varepsilon_{t-i} \tag{51}$$

Equations (49) and (51) follow from (48) and (50) respectively because $E_{t-1}\varepsilon_t = E_{t-1}\varepsilon_{t+1} = 0$.

Substituting (48)–(51) into (47):

$$\varepsilon_t - \left(1 + \frac{1}{\delta}\right) \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i} - \left(\alpha - \frac{1}{\delta}\right) \sum_{i=1}^{\infty} \pi_i \varepsilon_{t-i} + \alpha \sum_{i=1}^{\infty} \pi_{i+1} \varepsilon_{t-i} = 0 \tag{52}$$

Each ε_{t-i} can be any number so that (52) can hold if and only if the set of the coefficients on ϵ_t, on ϵ_{t-1}, on ϵ_{t-2} ...., each individually sums to zero. These sets must satisfy:

$$(\text{on } \varepsilon_t) 1 - \left(1 - \frac{1}{\delta}\right) \pi_0 = 0 \tag{53}$$

$$(\text{on } \varepsilon_{t-i}, i \geqslant 1) - (1+\alpha)\pi_i + \alpha \pi_{i+1} = 0 \tag{54}$$

Equation (54) is a homogeneous, difference equation in π_i with the same root as (27) above, and an analogous solution:

$$\pi_i = \pi_1 \left(\frac{1+\alpha}{\alpha}\right)^{i-1} \quad (i \geqslant 1) \tag{55}$$

Note that the forward root here is being 'driven backwards' artificially. In (55) again we see that there are an infinity of solutions chosen here by selecting π_1 arbitrarily and that only one is stable, namely that where $\pi_1 = 0$, which of course stops the forward root operating backwards in an unstable manner.

Invoking the stability condition we set $\pi_1 = 0$, so that $\pi_i = 0$ $(i \geqslant 1)$. From (53) we obtain $\pi_0 = \frac{\delta}{1+\delta}$. Our solution in p_t is therefore:

$$p_t = \overline{m} - y^* + \left(\frac{\delta}{1+\delta}\right) \epsilon_t \tag{56}$$

as before.

The Muth method becomes unwieldy for larger models where there are several errors like ε_t for each of which a sequence of coefficients must be determined, but it is often convenient for small illustrative models, and we shall use it frequently for this purpose.

Lucas Method of Undetermined Coefficients

A variant of the Muth method of undetermined coefficients has occasionally been used (for example, Barro, 1976; Lucas, 1972a) whereby the solution for the endogenous variables, instead of being written in terms of the constants and the errors, is written in terms of the 'state' variables, that is, current and past values of the exogenous variables (including the error terms of the model equations) and past values of the endogenous variables. (It therefore is like the Muth method in assuming that expected future exogenous variables can be reduced to current and past events; it, too, therefore drives the forward root backwards.) The need to include all the state variables can make this method unnecessarily complicated, as the example of this model shows.

Write the solution for p_t (on which we focus) as

$$p_t = \pi_1 \varepsilon_t + \pi_2 p_{t-1} + \pi_3 y_{t-1} + \pi_4 \varepsilon_{t-1} + \pi_5 \overline{m} + \pi_6 y^* \tag{57}$$

We have:

$$\overline{m} + \varepsilon_t = p_t + \frac{1}{\delta}(p_t - E_{t-1}p_t) + y^* - \alpha(E_{t-1}p_{t+1} - E_{t-1}p_t) \tag{58}$$

Use (57) to generate $E_{t-1}p_t$, $E_{t-1}p_{t+1}$ and substitute for these and p_t in (58), obtaining:

$$\begin{aligned}\overline{m} + \varepsilon_t = {} & \pi_t \varepsilon_t + \pi_2 p_{t-1} + \pi_3 y_{t-1} + \pi_4 \varepsilon_{t-1} + \pi_5 \overline{m} + \pi_6 y^* + \frac{1}{\delta}(\pi_1 \varepsilon_t) \\ & + y^* - \alpha[(\pi_2 - 1)(\pi_2 p_{t-1} + \pi_3 y_{t-1} + \pi_4 \epsilon_{t-1} + \pi_5 \overline{m} + \pi_6 y^*) \\ & + \pi_3 y^* + \pi_5 \overline{m} + \pi_6 y^*]\end{aligned} \tag{59}$$

We used $E_{t-1}y_t = y^*$ in this, from the Phillips curve. Now by the same argument as with the Muth method, the terms in each of the state variables must equate. So we have:

$$\text{(terms in } \varepsilon_t\text{): } 1 = \pi_1 + \frac{1}{\delta}\pi_1$$

yielding:

$$\pi_1 = \frac{1}{1 + \frac{1}{\delta}} = \frac{\delta}{1 + \delta} \tag{60}$$

$$\text{(terms in } p_{t-1}\text{) } 0 = \pi_2 - \alpha(\pi_2 - 1)\pi_2 = \pi_2(1 + \alpha) - \alpha\pi_2 \tag{61}$$

from which there are two solutions for $\pi_2 = 0$, $\frac{1+\alpha}{\alpha}$. Of these, $\frac{1+\alpha}{\alpha}$ (the forward root again being artificially driven backwards) violates the stability condition and is ruled out, leaving $\pi_2 = 0$.

$$\text{(terms in } y_{t-1}) \; 0 = \pi_3 - \alpha(\pi_2 - 1)\pi_3, \text{ implying } \pi_3 = 0 \tag{53}$$

$$\text{(terms in } \varepsilon_{t-1}) \; 0 = \pi_4 - \alpha(\pi_2 - 1)\pi_4, \text{ implying } \pi_4 = 0 \tag{54}$$

Given these solutions, the terms in $\overline{m}$ and y^* yield $\pi_5 = 1$, $\pi_6 = -1$. Hence we have obtained, if by a somewhat round-about route, the solution for p_t; that for y_t follows simply using the Phillips curve.

Clearly the method of solution is a matter purely of convenience. We have discussed several methods, all of which have been extensively used according to the problem and tastes of the problem solver. All have their advantages and disadvantages and are worth the reader's while to understand.

THE TECHNIQUES IN APPLICATION: A MORE COMPLICATED EXAMPLE

We now use a slightly more elaborate REFV model (with a unique stable solution) to illustrate further the application of these solution methods. The model here is 'fully dynamic', that is to say it returns to its steady state gradually after a shock rather than immediately as our previous models did. As such it is a prototype for many macro models used in practical analysis.

We retain our Cagan-style money demand equation but date the expectations at t for convenience in the money market[2]. We also retain our simple money supply equation (9); but we allow for adjustment costs in the response of output to unexpected price changes (our Phillips curve). So now we have a new model:

$$m_t = p_t + y_t - \alpha(E_t p_{t+1} - p_t) \; (\alpha > 0) \tag{62}$$

$$y_t - y^* = \frac{1}{\delta}(p_t - E_{t-1}p_t) + \mu(y_{t-1} - y^*) = \frac{1}{\delta}\frac{(p_t - E_{t-1}p_t)}{1 - \mu L} \tag{63}$$

$$m_t = \overline{m} + \varepsilon_t \tag{64}$$

where we have used the backward lag operator, L, in rewriting (63) to facilitate our subsequent operations.

[2] This dating of expectations implies that agents in the money and bonds markets have access to all current information whereas those in the goods and labour markets only have access to last period's — not a set-up with much theoretical appeal. Exactly what information which agents have is discussed carefully below (especially in the following chapters, 3 and 4). Here we make this assumption merely to illustrate our techniques with less complication.

Basic Method

Let us apply our adjusted method, focusing on the solution for p$_t$. Step 1 gives, substituting for (63) and (64) into (62):

$$\overline{m} + \varepsilon_t = (1+\alpha)p_t + y^* + \frac{1}{\delta}\frac{(p_t - E_{t-1}p_t)}{1-\mu L} - \alpha E_t p_{t+1} \qquad (65)$$

Rearranging and multiplying through by $(1-\mu L)$ yields:

$$(\overline{m} - y^*)(1-\mu) + \varepsilon_t - \mu\varepsilon_{t-1} = -\alpha E_t p_{t+1} + \left(1 + \alpha + \frac{1}{\delta}\right) p_t + \left(\alpha\mu - \frac{1}{\delta}\right) E_{t-1}p_t - (\mu + \alpha\mu)p_{t-1} \qquad (66)$$

Notice that the lag of $E_t p_{t+1}$ is $E_{t-1}p_t$ and not, for example, p_t or $E_{t-1}p_t$.

We now move to step $(2')$, where we must find $E_t p_{t+1}$ and $E_{t-1}p_{t+1}$. Accordingly, first we take expectations at $t = 1$ to obtain:

$$(\overline{m} - y^*)(1-\mu) - \mu\varepsilon_{t-1} = -\alpha E_{t-1}p_{t+1} + (1 + \alpha + \alpha\mu)E_{t-1}p_t - (\mu + \alpha\mu)p_{t-1} \qquad (67)$$

and:

$$(\overline{m} - y^*)(1-\mu) = -\alpha E_{t-1}p_{t+i+1} + (1+\alpha+\alpha\mu)E_{t-1}p_{t+i} - (\mu+\alpha\mu)E_{t-1}p_{t+i-1} \ (i \geqslant 1) \qquad (68)$$

The solution of (68) is:

$$E_{t-1}p_{t+i} = (\overline{m} - y^*) + A_1\left(\frac{1+\alpha}{\alpha}\right)^i + A_2\mu^i \ (i \geq 0) \qquad (69)$$

where A and B are determined by the initial values $E_{t-1}p_{t+1}$, $E_{t-1}p_t$. However, we have only one equation (67), to determine both $E_{t-1}p_{t+1}$ and $E_{t-1}p_t$, so that there is an infinity of paths, all but one unstable (this situation of a unique stable path from which movement in any direction is unstable is known as the 'saddlepath' property). Impose the stability condition, then set $A_1 = 0$ with the result that $A_2 = E_{t-1}p_t - (\overline{m} - y^*)$ so defining:

$$E_{t-1}p_{t+1} = \overline{m} - y^* + [E_{t-1}p_t - (\overline{m} - y^*)]\mu \qquad (70)$$

We can now use (67) to solve for $E_{t-1}p_t$ as:

$$E_{t-1}p_t = (\overline{m} - y^*)(1-\mu) - \frac{\mu}{1+\alpha}\varepsilon_{t-1} + \mu p_{t-1} \qquad (71)$$

We can infer immediately from (71) that:

$$E_t p_{t+1} = (\overline{m} - y^*)(1-\mu) - \frac{\mu}{1+\alpha}\varepsilon_t + \mu p_t \tag{72}$$

This can be verified by leading (67) one period, taking expectations at t, and repeating the operations in (68) to (71) but advanced one period.

We have now completed step $2'$ and proceed to step 3, substituting $E_{t-1}p_t$ and $E_t p_{t+1}$ from (71) and (72) into (66), to obtain after collecting terms:

$$p_t = (\overline{m} - y^*)(1-\mu) - \frac{\mu}{1+\alpha}\varepsilon_{t-1} + \mu p_{t-1} + \frac{1+\alpha-\alpha\mu}{(1+\alpha)(1+\alpha-\alpha\mu+\frac{1}{\delta})}\varepsilon_t \tag{73}$$

This model and its solution are illustrated in Figure 2.5. The initial shock to demand, ε_t, shifts the aggregate demand curve out to DD along the SS, short-run Phillips, curve. The position of DD takes account of $E_t p_{t+1}$, the expected value of next period's price level. This expectation solution is found by locating the unique stable path (the analogue of the algebra in equations (69) to (71)). The D^*D^* curve shows the combinations of (p, y) for which prices are not expected to change: the equation of the LM curve (62) is written as $E_t \Delta p_{t+i+1} = \frac{-1}{\alpha}(\overline{m} - E_t p_{t+i} - E_t y_{t+i}$ $(i \geqslant 1)$. The S^*S^* curve shows the combinations of (p, y) for which output is not expected to change: the Phillips curve, (63), is written as $E_t \triangle y_{t+i} = (\mu - 1)(E_t y_{t+i} - y^*)$ $(i \geqslant 1)$.

The arrows show the implied motion of (p, y) where they take values off these curves; the line with arrows pointing along it towards the steady-state equilibrium at the intersection of D^*D^* and S^*S^*, is the saddlepath, the unique stable solution. $E_t p_{t+1}$ jumps from p_t on to this line at the point where it intersects $S'S'$, the expected vertical Phillips curve given by the gradual adjustment of y_t back to y^*; going through this point accordingly in an aggregate demand curve, $D'D'$, whose equation is $\overline{m} - E_t y_{t+1} + \alpha E_t p_{t+2} = (1+\alpha)E_t p_{t+1}$. $D'D'$ looks forward to $E_t p_{t+2}$ which can be found in a similar way as the point on the saddlepath intersected by $E_t y_{t+2}$. Accordingly (p, y) are expected to converge at the rate μ on $(\overline{m} - y^*,\ y^*)$ along this saddlepath, after their initial shock to $(p_t,\ y_t)$.

The basic method using the Sargent forward operator approach

We proceed as above up to (68). Sargent now rewrites (68) as:

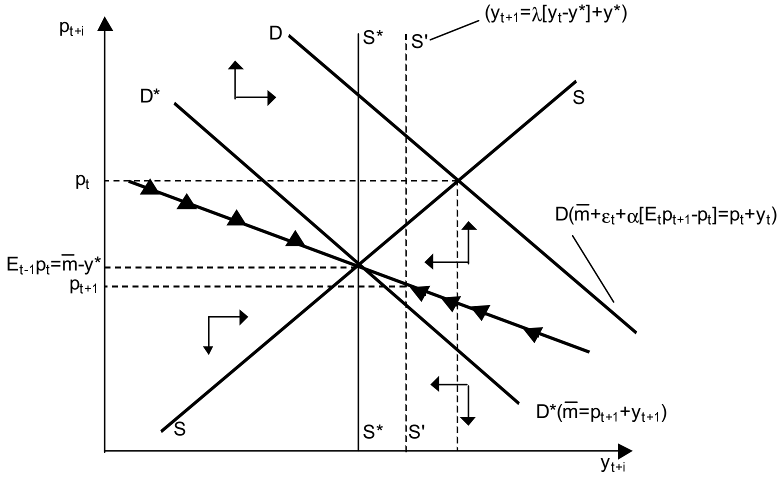


Figure 2.5: The solution of a fully dynamic model when a shock, ε_t, disturbs equilibrium

$$\frac{-1}{\alpha}(\overline{m}-y^*)(1-\mu) = \left[B^{-1} - \left(\frac{1+\alpha}{\alpha} + \mu\right) + \left(\frac{1+\alpha}{\alpha}\right)\mu B\right] E_{t-1}p_{t+i}$$
$$= \left(B^{-1} - \frac{1+\alpha}{\alpha}\right)(1-\mu B)E_{t-1}p_{t+i}$$
$$= -\frac{1+\alpha}{\alpha}\left[1 - \left(\frac{\alpha}{1+\alpha}\right)B^{-1}\right](1-\mu B)E_{t-1}p_{t+i} \quad (i \geqslant 1) \qquad (74)$$

Now we can write (74) as:

$$\left(\frac{1}{1+\alpha}\right)\frac{(\overline{m}-y^*)(1-\mu)}{\left(1-\frac{\alpha}{1+\alpha}B^{-1}\right)} = (1-\mu B)E_{t-1}p_{t+i} \quad (i \geqslant 1) \qquad (75)$$

If we impose stability, this yields setting $i = 1$:

$$E_{t-1}p_{t+1} = \mu E_{t-1}p_t + (\overline{m} - y^*)(1-\mu) \qquad (76)$$

which yields the rest of our solution as before.

The Sargent method thus represents a convenient extension of operator techniques to REFV models. ‘Backward roots’ (entering because of lagged adjustment) are projected backwards, that is, kept in the form $1/(1-\mu B)$, ‘forward roots’ (entering via expected future variables) are projected forwards, that is, transformed to $1/\left(1-\frac{\alpha}{1+\alpha}B^{-1}\right)$; this procedure, under the stability condition, gives us the same result as before, but in a very compact manner.

Sargent's method is particularly useful for dealing with delayed shocks which are nevertheless anticipated from a date before they occur; so far we have considered only contemporaneous, unanticipated shocks. But, for example, it may become known now that the government plans to raise the money supply sharply in two years' time for some reason to do with anticipated public finance difficulties.

To allow for such a possibility let us in (64) allow ε_t to be a shock which may be related to previous events, whereas before it was assumed to be unrelated. Now moving through the previous steps of our solution, we find that (66) is the same. Taking expectations at $t-1$, however, yields:

$$(\overline{m}-y^*)(1-\mu)+E_{t-1}\varepsilon_t-\mu\varepsilon_{t-1}=-\alpha E_{t-1}p_{t+1} \\ +(1+\alpha+\alpha\mu)E_{t-1}p_t-(\mu+\alpha\mu)p_{t-1} \quad (77)$$

and so:

$$(\overline{m}-y^*)(1-\mu)+E_{t-1}\varepsilon_{t+i}-\mu E_{t-1}\varepsilon_{t+i-1}=-\alpha E_{t-1}p_{t+i+1}+ \\ (1+\alpha+\alpha\mu)E_{t-1}p_{t+}i-(\mu+\alpha\mu)E_{t-1}p_{t+i-1} \quad (i\geq 1) \quad (78)$$

Sargent's (74) now becomes:

$$-\frac{1}{\alpha}(\overline{m}-y^*)(1-\mu)-\frac{1}{\alpha}(1-\mu B)E_{t-1}\varepsilon_{t+i}= \\ \left(1-\frac{1+\alpha}{\alpha}B\right)(1-\mu B)B^{-1}E_{t-1}p_{t+i} \quad (i\geqslant 1) \quad (79)$$

And (75):

$$(\overline{m}-y^*)(1-\mu)+\frac{1}{1+\alpha}\frac{1-\mu B}{1-(\frac{1+\alpha}{\alpha}B)^{-1}}E_{t-1}\varepsilon_{t+i} \\ =(1-\mu B)E_{t-1}p_{t+i} \quad (i\geq 1) \quad (80)$$

The left-hand side of this can be written:

$$(\overline{m}-y^*)(1-\mu)+\frac{1}{1+\alpha}\sum_{j=0}^{\infty}(\alpha/1+\alpha)'(E_{t-1}\varepsilon_{t+i+j}-\mu E_{t-1}\varepsilon_{t+i-1+j})$$

which implies for the case of $i=1$ the solution for:

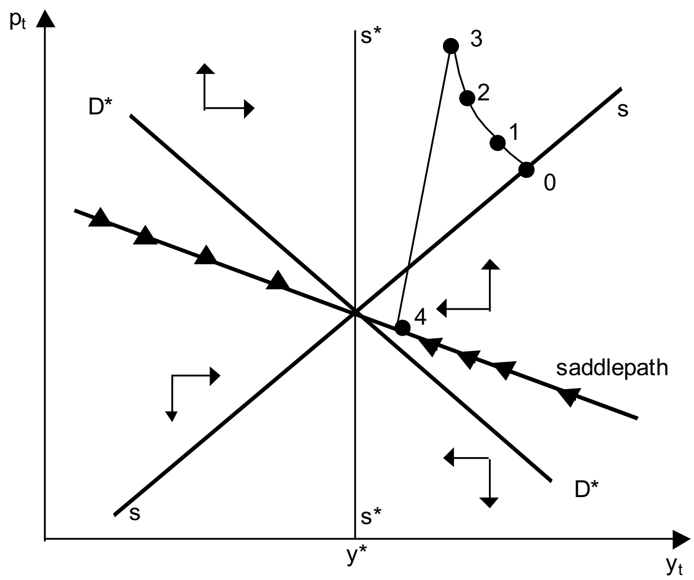
$$E_{t-1}p_{t+1}=\mu E_{t-1}p_t+(\overline{m}-y^*)(1-\mu)+1/(1+\alpha) \\ \sum_{j=0}^{\infty}[\alpha/(1+\alpha)]^j\left(E_{t-1}\varepsilon_{t+j+1}-\mu E_{t-1}\varepsilon_{t+j}\right) \quad (81)$$

We can also use (77) to solve for $E_{t-1}p_t$ as:

$$E_{t-1}p_t = (\overline{m} - y^*)(1-\mu) + \frac{1}{1+\alpha}E_{t-1}\varepsilon_t - \frac{\mu}{1+\alpha}\varepsilon_{t-1} + \mu p_{t-1} + \frac{1}{1+\alpha}\sum_{j=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^{j+1}(E_{t-1}\varepsilon_{t+j+1} - \mu E_{t-1}\varepsilon_{t+j}) \quad (82)$$

Now we see that the future shocks foreseen at $t-1$ for $t+j$ enter the expected solution for p_t with a coefficient of

$$\left[\frac{1}{1+\alpha}\right]\left[\left(\frac{\alpha}{1+\alpha}\right)^{j} - \mu\left(\frac{\alpha}{1+\alpha}\right)^{j+1}\right] = \left(\frac{\alpha}{1+\alpha}\right)^{j}\left(1-\mu\frac{\alpha}{1+\alpha}\right)\left(\frac{1}{1+\alpha}\right)$$



Shock ε_t occurs at *t=3*, anticipated at *t=0*. Numbers show the solution at each date, *t=0, 1, ...*

Figure 2.6: The effect of an unanticipated shock

Hence the forward root is 'thrown forwards', acting as a weight on the foreseen shock which diminishes the further ahead the shock occurs.

This is illustrated in Figure 2.6 for a positive demand shock anticipated at time t for three periods ahead. At t (= 0 on the figure) the expected future shock to demand raises prices unexpectedly, increases supply (a movement along SS), and stimulates demand (a shift in DD) because the rise in future prices relative to present prices reduces the demand for money, so increasing money expenditure. Demand continues to rise in $t = 1$ and $t = 2$ because future prices exceed current prices by a greater amount in $t = 1$ than $t = 0$ and in $t = 2$ than in $t = 1$; this is dictated by the dynamics (shown by the phase arrows) in that part of Figure 2.6. At $t = 3$, the shock occurs and increases demand further. Prices in $t = 4$ are expected to drop to exactly where they would have been (along the saddlepath) had the original shock to demand been an unanticipated one at $t = 0$ sufficient to stimulate output by the same amount as the anticipated shock did; so in $t = 3$, there are conflicting forces on demand, the positive effect of the $t = 3$ shock more than offsetting the effect from the expected future decline in prices. It is a useful exercise to assign numbers to the parameters and plot the resulting path as in figure 2.6.

Muth method

The Muth method, though not particularly intuitive, is probably the easiest to apply for this model. The general solution for p will be as before

$$p_t = \overline{p} + \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i} \tag{83}$$

and it remains that:

$$\overline{p} = \overline{m} - y^* \tag{84}$$

Box 2.1

The algebra for the illustration works out as follows if for convenience we set $E_{t-1}p_t = p_{t-1} = y_{t-1} = \varepsilon_{t-1} = \varepsilon_t = \overline{m} = y^* = 0$:

$$p_t = [\frac{\alpha}{1+\alpha+\frac{1}{\delta}}]E_t p_{t+1} \text{ from (65)}$$

$$E_t p_{t+1} = (\frac{1}{1+\alpha})(\frac{\alpha}{1+\alpha})^2(1-\mu\frac{\alpha}{1+\alpha})\varepsilon_{t+3} + \mu p_t \text{ from (82)}$$

led by one period and taking expectations at time t so that:

$$p_t = qx_t \text{ and } E_t p_{t+1} = (1+\mu q)x_t$$

where

$$\begin{aligned} x_t &= (\frac{1}{1+\alpha})(\frac{\alpha}{1+\alpha})^2(1-\mu\frac{\alpha}{1+\alpha})\varepsilon_{t+3}, \\ q &= \frac{\alpha}{1+\alpha-\alpha\mu+\frac{1}{\delta}} \end{aligned}$$

Again from (82)

$$\begin{aligned} E_t p_{t+2} &= \mu E_t p_{t+1} + \frac{1+\alpha}{\alpha}x_t = [\mu+\mu^2 q + \frac{1+\alpha}{\alpha}]x_t \\ E_t p_{t+3} &= \mu E_t p_{t+2} + \frac{1+\alpha}{\alpha}x_t = \\ & [\mu^2+\mu^3 q+\mu(\frac{1+\alpha}{\alpha})+(\frac{1+\alpha}{\alpha})^2]x_t \\ E_t p_{t+4} &= \mu E_t p_{t+3} - \mu\left(\frac{1}{1+\alpha}\right)\varepsilon_{t+3} \\ &= \mu\left\{ \begin{array}{c} \mu^2+\mu^3 q+\mu\left(\frac{1+\alpha}{\alpha}\right)+\left(\frac{1+\alpha}{\alpha}\right)^2 - \\ \left(\frac{1+\alpha}{\alpha}\right)^2\left(\frac{1+\alpha}{1+\alpha-\alpha\mu}\right) \end{array} \right\} x_t \\ E_t p_{t+5} &= \mu E t p_{t+4} \end{aligned}$$

Both $E_t p_{t+4}$ and $E_t p_{t+5}$ are negative.

We now substitute in (66) dropping constants to obtain the identities in the ε_{t-i} from:

$$\varepsilon_t - \mu\varepsilon_{t-1} = -\alpha\sum_{i=0}^{\infty}\pi_{i+1}\varepsilon_{t-1} + \left(1+\alpha+\frac{1}{\delta}\right)\sum_{i=0}^{\infty}\pi_i\varepsilon_{t-i} + \left(\alpha\mu-\frac{1}{\delta}\right)\sum_{i=1}^{\infty}\pi_t\varepsilon_{t-i} - (\mu+\alpha\mu)\sum_{i=1}^{\infty}\pi_{t-1}\varepsilon_{t-1} \quad (85)$$

The identities emerge as

$$(\varepsilon_t)\text{:}\quad 1 = -\alpha\pi_1 + \left(1+\alpha+\frac{1}{\delta}\right)\pi_0 \quad (86)$$

$$(\varepsilon_{t-1})\text{:}\quad -\mu = -\alpha\pi_2 + (1+\alpha+\alpha\mu)\pi_1 - (\mu+\alpha\mu)\pi_0 \quad (87)$$

$$(\varepsilon_{t-i}, i\geqslant 2)\text{:}\quad 0 = -\alpha\pi_{i+1} + (1+\alpha+\alpha\mu)\pi_i - (\alpha+\alpha\mu)\pi_{i-1} \quad (88)$$

Applying the stability condition to the solution of (88):

$$\pi_i = A_1\left(\frac{1+\alpha}{\alpha}\right)^{i-1} + A_2\mu^{i-1} \quad (i\geqslant 2) \quad (89)$$

sets $A = 0$, so that:

$$\pi_i = \pi_1 \mu^{i-1} \quad (i \geqslant 2) \tag{90}$$

Substituting this into (86) and (87) gives:

$$\pi_0 = \frac{1 + \alpha - \alpha\mu}{(1+\alpha)(1 + \alpha + \frac{1}{\delta} - \alpha\mu)} \tag{91}$$

$$\pi_1 = \frac{-\mu}{\delta(1+\alpha)(1 + \alpha + \frac{1}{\delta} - \alpha\mu)} \tag{92}$$

We can easily verify that this is the solution arrived at previously.

A MORE GENERAL WAY OF LOOKING AT AN REFV MODEL

We have been considering models with exogenous ('forcing') processes — here the money supply — that consist of a constant and a current shock. For most of the time we have assumed that the shock could not be predicted; here the unstable root was ruled out by our terminal condition. Then we looked at the case where one period's shock was known some periods in advance; here we showed that the unstable root determines how this shock works back to affect the present — in effect the root is 'thrown forward' and becomes stable when working backwards from the future. It is time to generalise the solution method we have been using to any sort of exogenous process.

To illustrate such a general method we take a variant of our earlier models:

$$m_t = p_t + y_t - \alpha(E_{t-1}p_{t-1} - E_{t-1}p_t) \tag{93}$$

$$y_t = y^* + \delta(p_t - E_{t-1}p_t) + \mu(y_{t-1} - y^*) \tag{94}$$

which will be recognised as a Cagan-style money demand function with inflation expectations dated at $t-1$ and a New Classical supply curve with persistence. Now let m_t be an exogenous process of a completely general sort: each period there is a new realization of m_t and a new set of $E_t m_{t+i} (i \geqslant 1)$. We will make no restrictions on how this set $(m_t,\ E_{t+i})$ changes at each t.

Consider the expectation at $t-1$ of this model:

$$E_{t-1}m_t = (1+\alpha)\left[1 - \frac{\alpha}{1+\alpha}B^{-1}\right]E_{t-1}p_t + E_{t-1}y_t \tag{95}$$

$$E_{t-1}y_t = y^* + \mu\delta\frac{(p_{t-1} - E_{t-1}p_{t-1})}{(1-\mu L)} \tag{96}$$

Let $p_t - E_{t-1}p_t = \eta_t$; we can easily solve for η_t as

$$\frac{1}{1+\delta}(m_t - E_{t-1}m_t) = \eta_t \tag{97}$$

by taking deviations from expected values across the model; this is a function of the purely unpredictable element (the innovation) in m_t.

Returning to our model above we can write the solution for $E_{t-1}p_t$ as:

$$\begin{aligned}E_{t-1}p_t = &\frac{E_{t-1}m_t}{(1+\alpha)(1-\frac{\alpha}{1+\alpha}B^{-1})} - y^* - \\ &\frac{\mu\delta\eta_{t-1}}{(1+\alpha)\left(1-\frac{\alpha}{1+\alpha}B^{-1}\right)(1-\mu L)} \\ &= \frac{1}{1+\alpha}\sum_{i=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^i (E_{t-1}m_{t+i} - E_{t-1}y_{t+i}) \end{aligned} \tag{98}$$

where

$$E_{t-1}y_{t+i} = y^* + \mu\delta\sum_{j=i}^{\infty}\mu^j\eta_{t-1-j+i} = y^* + \mu^{i+1}(y_{t-1} - y^*);$$

note that the expectations of future innovations are by definition zero.

Thus our solution automatically throws the term in α forwards (because it relates the present to expected future events) and the term in μ backwards (because it related the present and the expected future to past events). We can see that the general solution has a forward and a backward component, for each of which one of the roots of the model is appropriate.

The model requires, for stability, that both $\mid \frac{\alpha}{1+\alpha} \mid\prec 1$ and $\mid \mu \mid\prec 1$; both of these would be imposed as a matter of specification normally. Looking back over our previous discussion of uniqueness and will o' the wisp variables, we can also see that in these conditions the terminal condition will both ensure uniqueness and rule out bubbles.

Forward and backward roots: an examination

We can see that in the model we have been using — with slight differences in dating of the expectations — we have obtained an equation of the form:

$$... = \left[-\alpha B^{-1} + (1+\alpha+\alpha\mu) - (1+\alpha)\mu B\right] E_{t-1}p_{t+i} \tag{99}$$

If this is solved backwards we write it as:

$$... = -\alpha\left[1-\left(\frac{1+\alpha}{\alpha}+\mu\right))B-\left(\frac{1+\alpha}{\alpha}\right))\mu B^2\right]B^{-1}E_{t-1}p_{t+i} \quad (100)$$

The roots are then plainly $\frac{1+\alpha}{\alpha}$ and μ. These we recognise as the forward root, $\frac{\alpha}{1+\alpha}$, solved backwards and so inverted, and the backward root, μ.

If the difference equation is solved forwards we write it as:

$$... = -\mu(1+\alpha)\left[1-\left(\frac{\alpha}{1+\alpha}+\frac{1}{\mu}\right)B^{-1}-\left(\frac{\alpha}{1+\alpha}\right)\frac{1}{\mu}B^{-2}\right]BE_{t-1}p_{t+i} \quad (101)$$

The roots are then $\frac{1}{\mu}$ and $\frac{\alpha}{1+\alpha}$ which we recognise as the forward root solved forwards and the backward root, μ, solved forwards and so inverted.

Plainly to obtain the appropriate solution as permitted by the terminal condition, we solve the forward root forwards and the backward root backwards as we have seen, obtaining:

$$... = (1+\alpha)\left(1-\frac{\alpha}{1+\alpha}B^{-1}\right)(1-\mu B)E_{t-1}p_{t+i} \quad (102)$$

Equivalently we can obtain this appropriate solution by factorising

$$-\alpha B^{-1}+(1+\alpha+\alpha\mu)-(1+\alpha)\mu B = k_0(1-k_1B^{-1})(1-k_2B) = \\ -k_0k_1B^{-1}+k_0(1+k_1k_2)-k_0k_2B \quad (103)$$

where the undetermined coefficients are given by

$$k_0k_1=\alpha;\ k_0(1+k_1k_2)=1+\alpha+\alpha\mu;\ k_0k_2=\mu(1+\alpha) \quad (104)$$

Solving (96) for k_1 yields:

$$k_1^2-\left(\frac{\alpha}{1+\alpha}+\frac{1}{\mu}\right)k_1-\left(\frac{\alpha}{1+\alpha}\right)\frac{1}{\mu}=0 \quad (105)$$

(the characteristic equation of the model solved forward) whence $k_1 = \frac{\alpha}{1+\alpha},\ \frac{1}{\mu}$.

Alternatively solve (104) for k_2 to obtain:

$$k_2^2-(\frac{1+\alpha}{\alpha}+\mu)k_2-(\frac{1+\alpha}{\alpha})\mu=0 \quad (106)$$

(the characteristic equation solved backwards) yielding $k_2=\frac{1+\alpha}{\alpha},\ \mu$.

To find the stable solution we select the stable values of each root, forward (k_1), and backward (k_2).

STABILITY PROBLEMS IN RATIONAL EXPECTATIONS MODELS

When we considered adaptive expectations models we were concerned about whether they were stable or not. Clearly these were backward-looking models and this question amounted to whether the roots, all of them backward, were stable; if they were not, then we would naturally assume the specification was wrong, since we look for models that are stable, reflecting what we take to be a stable reality (if it were not, it should have exploded — yet it hasn't). With rational expectations models stability, as we have seen, involves in the two-root case both the forward and backward roots being stable when driven respectively forwards and backwards. If either is unstable, there is instability (either expected future events produce unstable current effects or past events produce unstable expected future effects). Let us consider them in turn: first, the case where the forward root is unstable — commonly known as the uniqueness problem.

The Uniqueness Problem: the unstable forward root

When the forward root is unstable, it is easiest to see what is happening in the simple model of equations (22), (8) and (9) at the beginning of the section on REFV models. Let us suppose that for some reason α in our model is negative and < -0.5. Suppose, for example, that there is a rigid relationship of money to average transactions in a period; and that precautionary transactions demand is positively related to the rate of inflation, because of the irregularity of price changes and the correlation between the size of these changes when they occur and the inflation rate (for example, I go to the doctor and find he had just put up his price by 30 per cent). This is implausible but not impossible.

Here we have, taking expectations at $t-1$, looking at the solution forwards:

$$\frac{\overline{m}-y^*}{1+\alpha} = \left(1 - \frac{\alpha}{1+\alpha}B^{-1}\right)E_{t-1}p_t \tag{107}$$

Plainly, the forward sum$(\bar{m}-y^*)/\left(1-\frac{\alpha}{1+\alpha}B^{-1}\right)$ does not converge. Equally if we look at the equation backwards we have:

$$-\frac{\overline{m}-y^*}{\alpha} = \left(1 - \frac{1+\alpha}{\alpha}B\right)E_{t-1}p_{t+1} \tag{108}$$

from which it follows that:

$$E_{t-1}p_{t+i} = \overline{m} - y^* + A\left(\frac{1+\alpha}{\alpha}\right)^i \quad (i \geqslant 1) \tag{109}$$

where $A = E_{t-1}p_t - (\overline{m} - y^*)$

This has a multiplicity of stable paths since $\left|\frac{1+\alpha}{\alpha}\right| < 1$: there is no unique stable path, hence the 'non-uniqueness' label of this case. Previously we used the stability condition to choose the unique stable path. However, now all the paths in (109) are stable, as shown in Figures 2.7 and 2.8, because we have rigged it so that $\left|\frac{1+\alpha}{\alpha}\right| < 1$. The stability condition is incapable of selecting a unique solution, therefore. This problem was first pointed out by Taylor (1977); and so far as we know there is nothing to rule out the possibility that REFV macroeconomic models will have an infinity of stable paths.

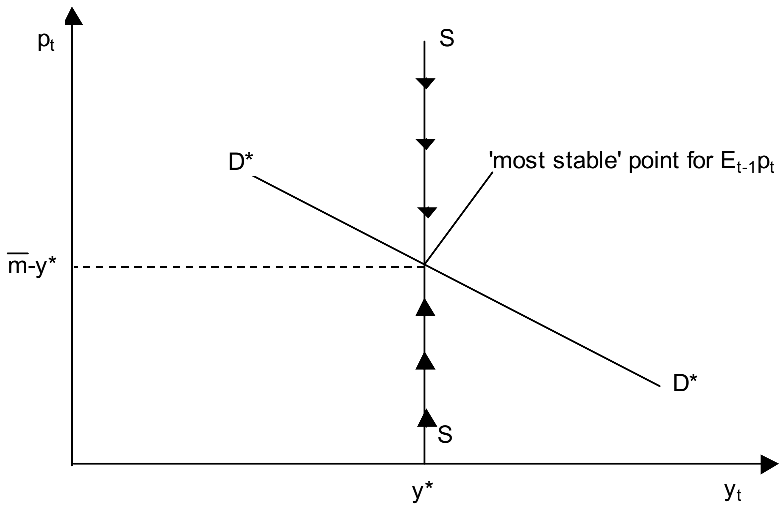


Figure 2.7: The uniquness problem in $(p_t, y_{t)}$ space

There is no generally agreed procedure among those using REFV models for this problem, other than to avoid using the ones with this property. One solution has, however, has been suggested by Minford et al. (1979) — to impose a terminal condition as we do in a normal model. Needless to say any 'solution' must somehow do violence to the model as specified since it is literally unstable. However, the economy may be, for some peculiar but genuine reason (as exemplified above), like this at least for a while: then could the terminal condition remove undesirable price level instability, as it did in the same model in its normal set-up? It turns out that it does indeed impose a unique stable solution. Thus we set:

$$E_{t-1}p_{t+N} = E_{t-1}p_{t+N+1} \qquad (110)$$

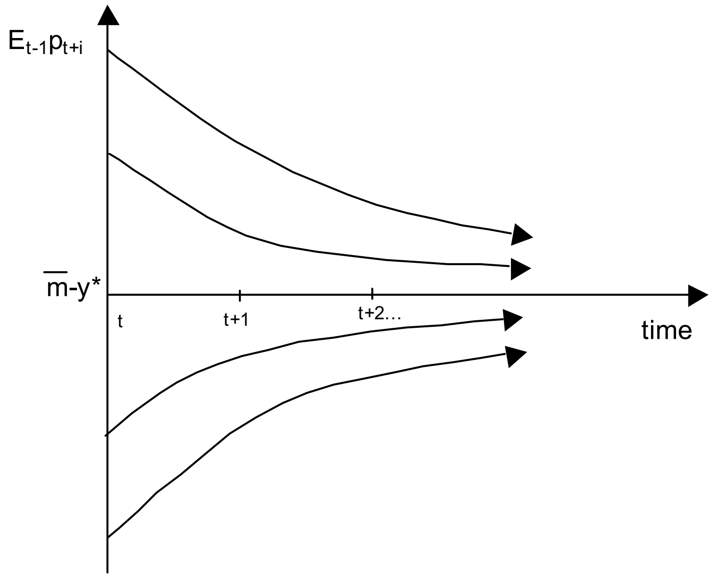


Figure 2.8: The uniqueness problem in (p_t, t) space

Using the forward solution we find that this implies

$$\frac{\overline{m} - y^*}{1+\alpha} = E_{t-1}p_{t+N} - \frac{\alpha}{1+\alpha}E_{t-1}p_{t+N+1} \tag{111}$$

whence $E_{t-1}p_{t+N} = \overline{m} - y^*$.

Via backwards recursion we obtain:

$$E_{t-1}p_{t+N-1} - \frac{1+\alpha}{\alpha}E_{t-1}p_{t+N} = \frac{\overline{m} - y^*}{1+\alpha} \tag{112}$$

whence $E_{t-1}p_{t+i} = \overline{m} - y^*$ $(i \geqslant 0)$.

Using the backwards solution, our terminal solution implies :

$$A\left(\frac{1+\alpha}{\alpha}\right)^{N+1} + \overline{m} - y^* = A\left(\frac{1+\alpha}{\alpha}\right)^{N} + \overline{m} - y^* \tag{113}$$

which is strictly valid only when $A = 0$. Thus also:

$$E_{t-1}p_{t+i} = \bar{m} - y^*(i \geqslant 0)$$

Note that just as in the case of our normal model, a bubble can be added to this solution, namely $(\frac{1+\alpha}{\alpha})^i E_{t-1}z_{t+i}$, but in this case it is an 'imploding bubble'. The terminal condition rules this out here, as it did the exploding bubble of our normal case. The justification for such a condition might seem strained in this case. Yet upon consideration it is equally justifiable. Non-uniqueness (forward instability and imploding bubbles) must cause quite as serious problems as backwards instability and exploding bubbles. For the endogenous variables may in each period jump by unpredictably large (strictly unbounded) amounts; even though they will subsequently be expected to return to equilibrium, in all subsequent periods there will be shocks with infinite variance. Such uncertainty would be likely to provoke changes in behaviour sufficient to create an incentive for the money issuer to make a commitment such as is set out in the terminal condition. This commitment would then limit the uncertainty as we have seen, to that associated with the 'most stable' path — a result much in the spirit of a suggestion by Taylor (1977) that the least variance path will be selected by 'collective rationality'[3].

Let us now apply this same approach to the more complicated model we used above:

$$m_t = p_t + y_t - \alpha(E_{t-1}p_{t+1} - E_{t-1}p_t) \tag{114}$$

$$y_t = y^* + \delta(p_t - E_{t-1}p_t) + \mu(y_{t-1} - y^*) \tag{115}$$

$$m_t = \bar{m} + \varepsilon_t \tag{116}$$

This yields a reduction in terms of prices and expected prices of:

$$\begin{aligned}(\overline{m} - y*)(1-\mu) + \varepsilon_t - \mu\varepsilon_{t-1} &= p_t - \mu p_{t-1} - \alpha E_{t-1}p_{t+1} + \alpha\mu E_{t-2}p_t \\ &+ \delta(p_t - E_{t-1}p_t) + \alpha E_{t-1}p_t - \alpha\mu E_{t-2}p_{t-1}\end{aligned} \tag{117}$$

Let us assume, as we shall see we must, that $\left|\frac{1+\alpha}{\alpha}\right| \succ |\mu|$. First solve the model forwards and impose the terminal condition to obtain:

$$E_{t-1}p_{t+N} = \overline{m} - y^* - \mu^{N+1}(y_{t-1} - y^*) \tag{118}$$

The backwards recursion proceeds:

$$E_{t-1}p_{t+N-1} = \frac{\alpha}{1+\alpha}E_{t-1}p_{t+N} + \frac{1}{1+\alpha}[\overline{m} - y^* - \mu^N(y_{t-1} - y^*)] \tag{119}$$

[3] There have been other suggestions, like Taylor's, as to how society would select such a path. Peel (1981) argues that the monetary authorities will select a feedback rule generating uniqueness; however, it is not clear that they do select such rules in practice. McCallum (1983) argues that the solution chosen, when framed according to the Lucas undetermined coefficients method, will contain only the minimum set of state variables (his MSV procedure). This, we would argue, would be a result of the sort of government commitment we refer to. In practical terms MSV and terminal conditions deliver the same solution.

ultimately reaching:

$$E_{t-1}p_t = \overline{m} - y^* - \left\{ \left(\frac{\alpha\mu}{1+\alpha}\right)^N + \frac{1}{1+\alpha}\left[\begin{array}{c} \left(\frac{\alpha\mu}{1+\alpha}\right)^{N-1} + \left(\frac{\alpha\mu}{1+\alpha}\right)^{N-2} + \ldots \\ + \frac{\alpha\mu}{1+\alpha} + 1 \end{array} \right] \right\} \mu(y_{t-1} - y^*) \quad (120)$$

which, letting N be large (where for convergence $\left|\frac{1+\alpha}{\alpha}\right| \succ |\mu|$)

$$\simeq \overline{m} - y^* - \frac{1}{1+\alpha(1-\mu)}\mu(y_{t-1} - y^*) \quad (121)$$

The same result is obtained using the backward solution; take expectations of (117) at $t-2$ to obtain:

$$\frac{-(1-\mu)}{\alpha}(\overline{m} - y^*) = E_{t-2}p_{t+1} - (\frac{1+\alpha}{\alpha} + \mu)E_{t-2}p_t + \mu(\frac{1+\alpha}{\alpha})E_{t-2}p_{t-1} \quad (122)$$

whose solution is:

$$E_{t-2}p_{t+i} = \overline{m} - y^* + A_1(\frac{1+\alpha}{\alpha})^{i+1} + A_2\mu^{i+1} \quad (i \geqslant 1) \quad (123)$$

with initial values $E_{t-2}p_t$ and $E_{t-2}p_{t-1}$.

The terminal condition forces the constant on the root with the highest modulus (that is, $\frac{1+\alpha}{\alpha}$) to be zero, whence:

$$E_{t-2}p_{t+i} = \overline{m} - y^* + [E_{t-2}p_{t-1} - (\overline{m} - y^*)]\mu^{i+1} \quad (i \geqslant 0) \quad (124)$$

whence also:

$$E_{t-1}p_{t+i+1} = \overline{m} - y^* + [E_{t-1}p_t - (\overline{m} - y^*)]\mu^{i+1} \quad (i \geqslant 0) \quad (125)$$

Substitute for $E_{t-1}p_{t+1}$ from (125) into (114), take expectations at $t-1$ of (114)–(116), and reduce to obtain:

$$(\overline{m} - y^*) = E_{t-1}p_t + \mu(y_{t-1} - y^*) - \alpha[(1-\mu)(\overline{m} - y^*) + \mu E_{t-1}p_t] \quad (126)$$

whence:

$$E_{t-1}p_t = \overline{m} - y^* - \frac{1}{1+\alpha(1-\mu)}\mu(y_{t-1} - y^*) \quad (127)$$

as with the forward solution.

There is no solution if $\left|\frac{1+\alpha}{\alpha}\right| \prec |\mu|$. (123) now sets $A_2 = 0$ which gives:

$$E_{t-1}p_{t+1} = (1 - \frac{1+\alpha}{\alpha})(\overline{m} - y^*) + \frac{1+\alpha}{\alpha}E_{t-1}p_t$$

Repeating the operations which gave (127), we now find that:

$$E_{t-1}p_t = \frac{\mu}{\alpha}(y_{t-1} - y^*)$$

implying:

$$E_{t-1}p_{t+1} = \frac{\mu}{\alpha}(E_{t-1}y_t - y^*) = \frac{\mu^2}{\alpha}(y_{t-1} - y^*)$$

When this is substituted again into (114)–(116) we obtain:

$$E_{t-1}p_t = \frac{\overline{m} - y^*}{1+\alpha} + \frac{\mu(1-\mu)}{1+\alpha}(y_{t-1} - y^*) \tag{128}$$

There is therefore no solution for $E_{t-1}p_t$ by contradiction. (If $\left|\frac{1+\alpha}{\alpha}\right| = |\mu|$, the terminal condition cannot impose a unique solution and (120) does not converge; so again there is no solution.)

What we have seen is that provided $\frac{\alpha}{1+\alpha}$ is not too unstable (ie. provided that $\left|\frac{\alpha}{1+\alpha}\right| \prec \left|\frac{1}{\mu}\right|$), the terminal condition will force a stable solution and rule out implosive bubbles.

The case of an unstable backward root

We can illustrate this case with the model we have just used. Here $|\mu| \succ 1$ in which case output is expected to explode — clearly an inadmissible model in general since the backward sums $\delta\varepsilon_t/\left[(1+\delta)(1-\mu L)\right]$ do not converge.

In this case, too, however, there is a reason for seeking some sort of a solution. Such a model cannot be ruled out for an episode (for example moving between two stable models in a general non-linear model — as discussed in the supply-side chapter *a propos* of virtuous and vicious circles and more generally in the Time-Series Annex). We therefore need to face up to the problems for inflation and monetary policy in such an episode. We have justified our terminal condition as a restriction placed on behaviour by the monetary authorities to prevent undesirable price outcomes. So here we ask if a terminal condition will produce acceptable price behaviour.

We can in fact simply use our workings in the previous section, while noting here that $\alpha/(1+\alpha) \prec 1$. Again we require for a solution that

$(1+\alpha)/\alpha \succ |\mu|$: that is, that here the backward root not be too unstable. Thus if $\alpha/(1+\alpha) \succ 1$ (our last case of an unstable forward root) then $(1+\alpha)/\alpha \prec 1 \prec |\mu|$ and so there can be no solution with a terminal condition.

However with $(1+\alpha)/\alpha \succ |\mu|$ we obtain the same solution as above, namely:

$$E_{t-1}p_{t+i+} = \overline{m} - y^* - \mu^{i+1}\frac{1}{1+\alpha(1-\mu)}(y_{t-1} - y^*) \quad (i \geqslant 0) \qquad (129)$$

This shows clearly that a terminal condition will solve for a price path, that it will be unique, but that it will be unstable matching the instability in output — we might label this as 'controlled price instability'.

What we have shown about models with stability problems is that, by introducing a terminal condition justified by monetary policy reactions, we will in many cases — where the instability is not too severe — find unique solutions: if the instability is too severe, there will be no solution at all under a terminal condition. In our particular model with two roots, 'not too severe' means $(1+\alpha)/\alpha \succ |\mu|$. This implies that if both forward and backward roots are unstable there is no solution, since $(1+\alpha)/\alpha \prec 1 \prec |\mu|$. More general models with more roots have to be examined case by case if they have instability of either sort, using the analytical techniques we have described in this chapter.

CONCLUSIONS

This has been a chapter designed to equip the reader with the techniques to solve rational expectations models in a manner useful to applied work[4]. We have shown how to use four main methods of solution: a basic method, both with the Sargent forward operator and with the model solved backwards, and the Muth and Lucas undetermined coefficients methods. We have also discussed the criterion for choosing a unique solution in these models, free of extraneous or 'will o' the wisp' variables. The criterion we propose, namely that terminal conditions are imposed on the model (some external 'transversality condition'), is widely accepted in practice The effect of this condition is to ensure a stable path free of extraneous variables or bubbles. Practical methods

[4] There are a number of descriptions of solution methods available in the literature (see, e.g., Shiller 1978; and the useful Aoki and Canzoneri, 1979). For more complex applications than those considered in this chapter, the reader will invariably use numerical methods on the computer.

of solution used vary (see for example Wallis et al., 1985). However all of them are approximations to the analytic bubble-free solution we have been setting out above.

APPENDIX 2A: WHEN ADAPTIVE EXPECTATIONS ARE RATIONAL

Suppose that a series is generated by the ARIMA(0, 1, 1) process (see time-series annex at the end of the book):

$$y_t = y_{t-1} + u_t - ju_{t-1} \tag{1}$$

where u_t is a Gaussian white noise process and j a positive constant.

The rational expectation $E_{t-1}y_t$ of (1) is given by:

$$E_{t-1}y_t = y_{t-1} - ju_{t-1} \tag{2}$$

From (1):

$$y_t - y_{t-1} = (1 - jL)u_t \tag{3}$$

where L is the lag operator.

Substituting for u_{t-1} from (3) into (2) we obtain:

$$E_{t-1}y_t = y_{t-1} - j(\frac{y_{t-1} - y_{t-2}}{1 - jL})$$

so that:

$$E_{t-1}y_t - jE_{t-2}y_{t-1} = y_{t-1} - jy_{t-2} - j(y_{t-1} - y_{t-2}) \tag{4}$$

Subtracting $E_{t-2}y_{t-1}$ from both sides of (4) and rearranging we obtain

$$E_{t-1}y_t - E_{t-2}y_{t-1} = \kappa\{y_{t-1} - E_{t-2}y_{t-1}\} \tag{5}$$

where $\kappa = 1 - j$

Notice also from (5) that:

$$E_{t-1}y_t = \frac{\kappa y_{t-1}}{(1 - (1 - \kappa)L)} = \frac{(1 - \lambda)y_{t-1}}{(1 - \lambda L)}$$
$$= (1 - \lambda)(1 + \lambda L + \lambda^2 L^2 + \lambda^3 L^3 + \ldots +)y_{t-1} \tag{6}$$

where $\lambda = 1 - \kappa$ so that $\lambda = j$.

Equation (5) will be recognised as an adaptive expectations process. In other words if a variable is described by an ARIMA(0, 1, 1) process then adaptive expectations can be rational expectations if the coefficient of adaptation is equal to $1 - j$.

More generally we should note that a variety of mechanistic forms of expectation formation can be rational in a particular model structure. For instance regressive expectations are rational in the Dornbusch overshooting model considered in Chapter 14. The point is, of course, that these mechanistic expectations mechanisms will, in general, cease to be rational if policy regimes change.

Two interesting examples where adaptive expectations are rational are in models proposed by Sargent and Wallace (1973) and Muth (1961).

The Sargent–Wallace model of a hyperinflation

The demand for real balances has the form:

$$\log\left(\frac{M_t}{P_t}\right) = \alpha\pi_t^e + \gamma Y + \varphi + u_t \quad \alpha < 0, \quad \gamma > 0. \tag{6}$$

where M is the demand for nominal balances (assumed equal to supply), P is the price level, π_t^e is the public's expectation of future inflation, so $\pi_t^e = E_t \log P_{t+1} - \log P_t$, is assumed known. Y is real income assumed constant and u_t is a stochastic error term with an average value of zero. $\alpha,\ \gamma$ and φ are parameters.

Taking the first difference of (6) we obtain

$$\mu_t = \pi_t + \alpha(\pi_t^e - \pi_{t-1}^e) \tag{7}$$

where $\mu_t = \log\left(\frac{M_t}{M_{t-1}}\right)$ is the rate of change of the money supply and $\pi_t = \log\left(\frac{P_t}{P_{t-1}}\right)$ is the rate of inflation. It is assumed that $u_t - u_{t-1} = \eta_t$ where η_t is a white noise error.

Adaptive expectations (here current expectations of future inflation) can be written here as:

$$\pi_t^e = \frac{(1-\lambda)\pi_t}{1-\lambda L} \tag{8}$$

Substituting (8) into (7) we obtain the solution for π_t as:

$$\{[1+\alpha(1-\lambda)] - [\lambda + \alpha(1-\lambda)]L\}\pi_t = (1-\lambda L)\mu_t - (1-\lambda L)(1-L)u_t \tag{9}$$

so that the current inflation rate is determined by distributed lags of changes in the money supply and of the disturbance in the demand function.

To provide a rationalization of how adaptive expectations could have the rationality property, suppose the rate of monetary expansion is governed by the process:

$$\mu_t = \frac{(1-\lambda)\pi_t}{1-\lambda L} + \varepsilon_t \tag{10}$$

where ε_t is a white noise disturbance.

Substitute from (10) into (9) for μ_t to obtain

$$\{[\lambda + \alpha(1-\lambda)] - [\lambda + \alpha(1-\lambda)]L\}\pi_t = (1-\lambda L)\left[\varepsilon_t - (u_t - u_{t-1})\right] \tag{11}$$

Equation (11) can be rewritten as

$$[\lambda + \alpha(1-\lambda)]\,(1-L)\,\pi_t = (1-\lambda L)\,[\varepsilon_t - (u_t - u_{t-1})] \tag{12}$$

Now $\frac{1-L}{1-\lambda L} \equiv 1 - \frac{(1-\lambda)L}{1-\lambda L}$ so that (12) can be rewritten as:

$$\pi_t = \frac{(1-\lambda)\pi_{t-1}}{1-\lambda L} + [\lambda + \alpha(1-\lambda)]^{-1}\,[\varepsilon_t - (u_t - u_{t-1})] \tag{13}$$

Recalling that $u_t - u_{t-1} = \eta_t$ the rational expectation of (13) is given by:

$$E_{t-1}\pi_t = \frac{(1-\lambda)\pi_{t-1}}{1-\lambda L} \text{ implying that } E_t\pi_{t+1} = \pi_t^e = \frac{(1-\lambda)\pi_t}{1-\lambda L} \tag{14}$$

In other words if the money supply process is (10) the adaptive expectations will be rational. The question is why should the money supply process follow (10) in a hyperinflation. It is typically assumed that in a hyperinflation period the authorities print money to finance their nominal expenditure, assuming the level of real government expenditure, G, is constant. This assumption is captured in continuous time (where $\dot{x} = \frac{\partial x}{\partial t}$) by:

$$\frac{\dot{M}(t)}{P(t)} = G$$

or

$$\frac{\dot{M}}{M} = \frac{PG}{M} \tag{15}$$

In continuous time the demand for real balances is given by:

$$\frac{M}{P} = f\,[\pi^e(t)] \tag{16}$$

so that substituting (16) into (15) gives

$$\frac{\dot{M}}{M} = \frac{G}{f(\pi^e)} \tag{17}$$

Assuming a unique solution to (17) a linear discrete-time approximation to (17) is given by

$$\mu_t = E_t\pi_{t+1} + \varepsilon_t \tag{18}$$

where ε_t is assumed to be a white noise error term.

Assuming rational expectations (10) is equivalent with (18) (since $E_t\pi_{t+1} = \frac{(1-\lambda)\pi_t}{1-\lambda L}$).

Consequently Sargent and Wallace have demonstated how adaptive expectations can be rational in a hyperinflationary period.

We also note from (7) and (18) that

$$E_t\pi_{t+1} + \epsilon_t = \pi_t + \alpha(\pi_t^e - \pi_{t-1}^e) + \phi_t \tag{19}$$

or

$$\pi_t^e = \pi_t + \alpha(\pi_t^e - \pi_{t-1}^e) + \phi_t - \varepsilon_t \tag{20}$$

If expectations are adaptive:

$$\pi_t^e = \frac{(1-\lambda)\pi_t}{1-\lambda L} \tag{8}$$

Substituting (8) into (20) for $\pi_t^e,\ \pi_{t-1}^e$ we obtain

$$\frac{(1-\lambda)\pi_t}{1-\lambda L} = \pi_t + \alpha(1-L)\frac{(1-\lambda)\pi_t}{1-\lambda L} + \phi_t - \varepsilon_t \tag{21}$$

Simplifying (21) we obtain

$$\pi_t - \pi_{t-1} = -\left[(\lambda + \alpha(1-\lambda)^{-1}(1-\lambda L)(v_t)\right] \tag{22}$$

where v_t is the composite white noise error, $v_t = \phi_t - \varepsilon_t$

The process (22) is an ARIMA(0, 1, 1) as required for adaptive expectations of inflation to be rational.

Agricultural prices

The adaptive expectations assumption was widely employed in modelling of agricultural markets (see Nerlove, 1958) since models embodying this assumption were readily able to generate stable cyclical fluctuations. It is useful to illustrate how an agricultural model embodying rational expectations can also readily exhibit such fluctuations in price or how price series for storable commodities can approximately follow a random walk.

We employ the model of Muth (1961). We write the model as follows (all means are put to zero for simplicity):

$$q_t^d = -\beta p_t \tag{23}$$

$$q_t^s = \gamma E_{t-1}p_t + u_t \tag{24}$$

$$I_t = \alpha(E_t p_{t+1} - p_t) \tag{25}$$

$$q_t^d + \rho I_t = q_t^s + \rho I_{t-1} \tag{26}$$

where q_t^d, q_t^s are quantity demanded and supplied, respectively, p_t is price, I_t is a speculative inventory and u_t is an error process; α, β, γ and ρ are constants. In this model it is assumed that storage of a commodity is possible, and that storage, transactions costs and interest rates are negligible. Consequently, a speculative inventory exists which depends on the anticipated capital gain from holding the stock. The parameter α, which measures the response of inventory demands to expected price changes, is a function of the degree of risk aversion and the conditional variance of prices in Muth's exposition (also see Turnovsky, 1983).

Because storage can occur, equilibrium does not require that current production (supply) equals current consumption demand. Equation (26) represents the market equilibrium condition. A parameter ρ (= 1 or 0) is introduced for analytical convenience: if storage occurs it is equal to unity; otherwise if we set $\rho = 0$ and $\alpha = 0$, we have the standard, no storage, market clearing model.

Substitution of equations (23), (24) and (25) into (26) yields the reduced form:

$$-\beta p_t + \rho\alpha(E_t p_{t+1} - p_t) = \gamma E_{t-1} p_t + \rho\alpha(E_{t-1} p_t - p_{t-1}) + u_t \qquad (27)$$

Assume initially that u_t is white noise. Solve this model under rational expectations using the Lucas method of undetermined coefficients.

If we let

$$p_t = a p_{t-1} + b u_t \qquad (28)$$

for $\rho = 1$ we obtain by substitution

$$b = \frac{1}{a\alpha - (\alpha + \beta)} \text{ and } a = 1 + 0.5\left[\frac{(\beta+\gamma)}{\alpha}\right] - 0.5\left\{\left[2 + \frac{(\beta+\gamma)}{\alpha}\right]^2 - 4\right\}^{0.5} \qquad (29)$$

with $0 < a < 1$.

The solution for prices (28) is of interest. Price exhibits serially correlated fluctuations around its mean value when u_t, a variable which represents exogenous influences, is random. The reason for this is that inventories smooth out the effects of disturbances (shocks) to demand or supply. Consider, for instance, an abnormally good harvest due to favourable weather. In a market without storage, the additional supply will impact on market price in the current period. However with an inventory demand, speculators will buy some of the harvest, since the price in the future will, *ceteris paribus*, be greater than today, as weather returns to its normal expected value. This procedure will generally dampen price fluctuations. In addition the shock in the current period will have an impact in future periods which in this context is another way of saying that price movements will exhibit serial correlation.

Equation (28) also leads to another insight, as pointed out by Muth. As the importance of inventory speculative demands dominate a market relative to flow demands or supplies (as is likely over short periods of time) and consequently α becomes large relative to β or γ then (29) implies that a will become close to one. Consequently, price will approximate a random walk. This is an empirical feature often noted in price series for storable commodities in high frequency data. We also note that in this model the rational expectation of price is a fixed multiple of last period's price, though the coefficient is a function of the parameters of the model. In general, rational price expectations will be a function of lagged prices, though not in a mechanistic fixed fashion as occurs in an adaptive expectations model. Muth, however, did use a special case of the above model to illustrate how an adaptive expectations scheme could be rational. If we let $\rho = 0$ and $\alpha = 0$ we obtain the reduced form of the market clearing model given by

$$-\beta p_t = \gamma E_{t-1} p_t + u_t \tag{30}$$

Suppose that the error process is serially correlated so that

$$u_t = u_{t-1} + v_t \tag{31}$$

where v_t is white noise.

Consequently the rational expectation of (30) is given by

$$-(\beta + \gamma) E_{t-1} p_t = u_{t-1} \tag{32}$$

If we lag (30) one period and substitute for u_{t-1} in (32) we obtain

$$-(\beta + \gamma) E_{t-1} p_t = -\beta p_{t-1} - \gamma E_{t-2} p_{t-1} \tag{33}$$

Adding $(\beta + \gamma) E_{t-2} p_{t-1}$ to the left- and right-hand side of (33) we obtain after simplification the adaptive expectations scheme

$$E_{t-1} p_t - E_{t-2} p_{t-1} = \frac{\beta}{\beta + \gamma} (p_{t-1} - E_{t-2} p_{t-1}) \tag{34}$$

Notice from (30) and (32) that

$$-\beta p_t = -\frac{\gamma u_{t-1}}{\beta + \gamma} + u_t = \left(1 - \frac{\gamma L}{\beta + \gamma}\right) u_t \tag{35}$$

Differencing (35) and substituing for u_t from (31) we obtain

$$p_t - p_{t-1} = -\beta^{-1} \left(1 - \frac{\gamma L}{\beta + \gamma}\right) v_t \tag{36}$$

so that as required prices follow an ARIMA(0, 1, 1) process.

Finally it is interesting to consider the properties of a simple adaptive scheme when it is not rational. Consider the Wold decomposition for a variable

$$p_t = \overline{p}_t + \sum_{i=0}^{\infty} \pi_i u_t \tag{37}$$

where $\overline{p}_t$ is the mean component which could include a deterministic trend.

For ease of exposition consider the simplest adaptive expectation, namely

$$p_t^e = p_{t-1} = \overline{p}_{t-1} + \sum_{i=0}^{\infty} \pi_i u_{t-i} \tag{38}$$

The forecast error is given by

$$p_t - E_{t-1}p_t = \overline{p}_t + \sum_{i=0}^{\infty} \pi_i u_t - (\overline{p}_{t-1} + \sum_{i=0}^{\infty} \pi_i u_{t-i}) \tag{39}$$

In the case where $\overline{p}$ is a constant so $\overline{p}_t = \overline{p}_{t-1}$, we observe that the adaptive forecast is unbiased, since the average value of the forecast error is zero. However the forecast is inefficient in general since the right-hand side of (39) being a function of past information implies that the forecast error is correlated with information known at the time expectations were formed. If $\overline{p}_t$ contains a deterministic trend so that $\overline{p}_t = a + bt$, then $\overline{p}_t - \overline{p}_{t-1} = b$ so that from (39) the adaptive forecast will exhibit systematic bias and inefficiency.