Linear RE Models: The simplest example

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Summary

1. The simplest possible model
2. Two types of models the we may find in modern macroeconomics
3. The New Keynesian Model in the Blanchark-Khan form

I – The simplest possible model

The simplest possible model

1. Consider a model with a forward looking variable (y_t), a predetermined variable (x_t), two constants (α, ϕ), and a component representing exogenous shocks (v_t)

$$y_t = \alpha + \beta E_t y_{t+1} + x_t$$

$$x_t = \phi + \lambda x_{t-1} + v_t$$

$$v_t \sim iid(0, \sigma^2)$$

2. This simple model is very easy to solve if we impose two conditions on the solution:
    1. If the **law of iterated expectation** holds
    2. If "certainty equivalence" holds: the optimal (or true) values of y_t and x_t are the same as if we knew y_t and x_t for certain (if there were no uncertainty).

Solution by iterating forward: forward looking variable

1. The solution to the forward looking variable at the nth iteration will be

$$y_t = (\beta)^n E_t y_{t+n} + \sum_{i=0}^{n-1} (\beta)^i \alpha + \sum_{i=0}^{n-1} (\beta)^i E_t x_{t+i}$$

2. Avoiding explosive behavior of y_t when $n \longrightarrow \infty$, we have to impose

$$|\beta| < 1, \quad \text{or} \quad |1/\beta| > 1$$

3. Then we obtain

$$y_t = \sum_{i=0}^{n-1} (\beta)^i \alpha + \sum_{i=0}^{n-1} (\beta)^i E_t x_{t+i} \tag{1}$$

4. Therefore the optimal/true value of y_t depends only upon the value of a constant and the the expected values of the predetermined variable

Solution by iterating forward: predetermined variable

1. The solution to the backward looking variable at the nth forward iteration will be

$$x_t = (\lambda)^n x_0 + \sum_{i=0}^{n-1} (\lambda)^i \phi + \sum_{i=0}^{n-1} (\lambda)^i v_{t-i}$$

2. Avoiding explosive behavior of x_t when $n \to \infty$, we have to impose

$$|\lambda| < 1$$

3. And get

$$x_t = \sum_{i=0}^{n-1} (\lambda)^i \phi + \sum_{i=0}^{n-1} (\lambda)^i v_{t-i}$$

4. Assuming certainty equivalence

$$x_t = \sum_{i=0}^{n-1} (\lambda)^i \phi = \frac{\phi}{1-\lambda}$$

Solution by iterating forward: the entire model

1. Knowing the equilibrium level of x_t

$$x_t = \bar{x} = \frac{\phi}{1-\lambda}$$

2. Plugging it into the equilibrium level of y_t

$$y_t = \sum_{i=0}^{n-1} (\beta)^i \alpha + \sum_{i=0}^{n-1} (\beta)^i E_t x_{t+i}$$

3. Leads to

$$\begin{aligned} y_t &= \bar{y} = \frac{\alpha}{1-\beta} + \sum_{i=0}^{n-1} \beta^i \bar{x} = \frac{\alpha}{1-\beta} + \frac{1}{1-\beta}\bar{x} \\ &= \frac{\alpha + \phi/(1-\lambda)}{1-\beta} \end{aligned}$$

Numerical simulation

1. Now consider that the shock follows an AR(1) process

$$v_t = \rho v_{t-1} + \varepsilon_t, \quad |\rho| < 1$$

with

$$\varepsilon_t \sim iid(0, \sigma^2)$$

2. What happens to the whole model?
3. Parameters

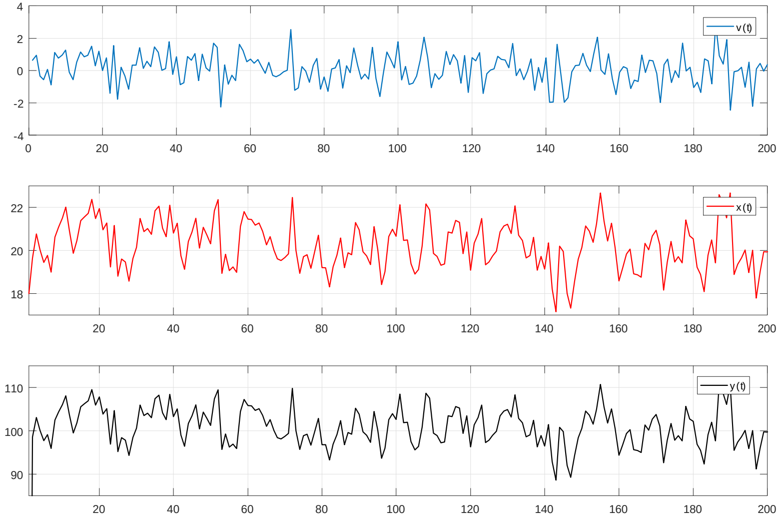
$$\beta = 0.75, \;\; \phi = 10, \;\; \lambda = 0.5, \;\; \alpha = 5, \;\; \rho = 0.8$$

1. Deterministic steady state

$$\bar{x} = 20 \;\; ; \;\; \bar{y} = 100$$

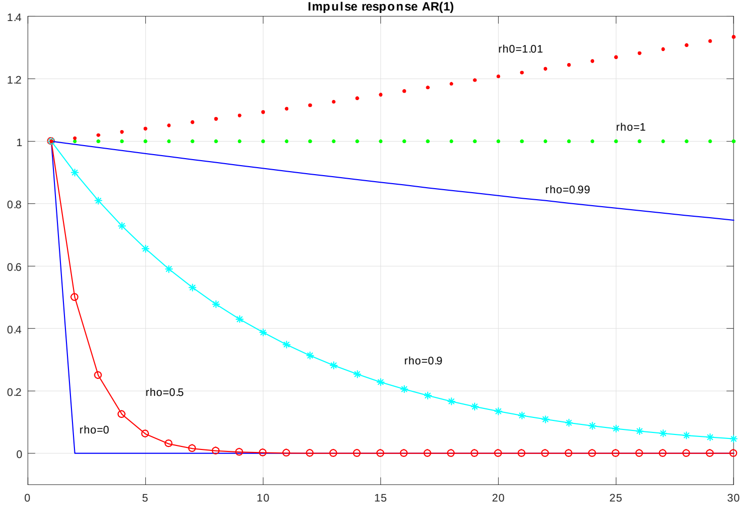
2. Time series and impulse response functions (next figures)

Time series

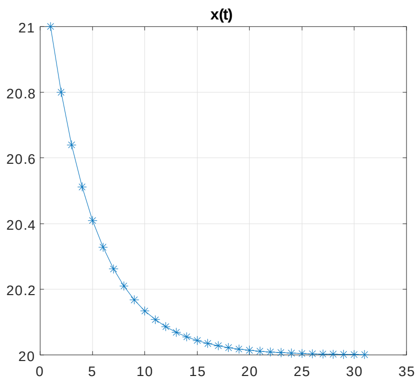


v(t)
x(t)
y(t)
4
2
0
-2
-4
22
20
18
110
100
90
0
20
40
60
80
100
120
140
160
180
200

Imp. response functions v(t): different values for rho



Impulse response functions: rho=0.8



The fundamental strategy to solve DSGE Models

1. Iterate forward and guarantee stability:
    1. Forward looking variables (also known as "control" or "jump" variables): $|1/\beta| > 1$
    2. Backward looking or predetermined variables: $|\lambda| < 1$
2. Solve the predetermined block
3. Then insert this solution into the forward looking block
4. We get a solution like this

$v(t)$

$x(t-1)$ $\quad x(t) \quad y(t)$

II – More complicated types of models

The New Keynesian Model: one version

1. This version includes five equations:

$$
\begin{aligned}
x_t &= E_t x_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1}\right) && \text{(IS)} \\
\pi_t &= \beta \cdot E_t \pi_{t+1} + \kappa x_t + u_t && \text{(AS)} \\
i_t &= \delta \pi_t + v_t && \text{(Rule)} \\
v_{t+1} &= \rho_v v_t + \epsilon_{t+1}^{v} && \text{(Shock v)} \\
u_{t+1} &= \rho_u u_t + \epsilon_{t+1}^{u} && \text{(Shock u)}
\end{aligned}
$$

2. The model is linear
3. Can we apply the same strategy? Yes!

The New Keynesian Model: onother version

1. The baseline version includes five equations:

$$
\begin{aligned}
x_t &= E_t x_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1} - r_t^n\right) && \text{(IS)} \\
\pi_t &= \beta \cdot E_t \pi_{t+1} + \kappa x_t && \text{(AS)} \\
i_t &= \phi_\pi \pi_t + \phi_x x_t && \text{(Monetary policy rule)} \\
r_t^n &= \rho r_{t-1}^n + \epsilon_t && \text{(Natural real interest rate)}
\end{aligned}
$$

2. The model is linear
3. Can we apply the same strategy? Yes!

The simple RBC model

1. The baseline version includes seven equations:

$$
\begin{aligned}
R_{t+1} &\equiv \alpha\left(Y_{t+1}/K_t\right)+1-\delta && \text{(S1)}\\
C_t^{-\eta} &= \beta E_t(C_{t+1}^{-\eta}R_{t+1}) && \text{(S2)}\\
Y_t/N_t &= \left[\xi/\left(1-\alpha\right)\right]C_t^{\eta} && \text{(S3)}\\
K_t &= (1-\delta)K_{t-1}+I_t && \text{(S4)}\\
Y_t &= A_t K_{t-1}^{\alpha} N_t^{1-\alpha} && \text{(S5)}\\
C_t+I_t &= Y_t && \text{(S6)}\\
\ln A_t &= (1-\rho)\ln A^* + \rho \ln A_{t-1} + \varepsilon_t && \text{(S7)}
\end{aligned}
$$

2. **A large nonlinear system** of stochastic difference equations
3. **Closed form solution is not possible** to be obtained for this model
4. Can we apply the strategy presented above in this case? Yes, if we linearize the model.

III – The NKM in the Blanchard-Kahn framework

The NKM in the Blanchard-Kahn framework

1. The baseline version includes five equations:

$$\begin{aligned} x_t &= E_t x_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1}\right) && \text{(IS)} \\ \pi_t &= \beta \cdot E_t \pi_{t+1} + \kappa x_t + u_t && \text{(AS)} \\ i_t &= \delta \pi_t + v_t && \text{(Rule)} \\ v_{t+1} &= \rho_v v_t + \epsilon^v_{t+1} && \text{(Shock v)} \\ u_{t+1} &= \rho_u u_t + \epsilon^u_{t+1} && \text{(Shock u)} \end{aligned}$$

2. Can we apply the same strategy?

$$\begin{aligned} \text{Predetermined} &: \quad v_t, u_t \\ \text{Forward looking} &: \quad x_t, \pi_t \end{aligned}$$

The NKM in state space

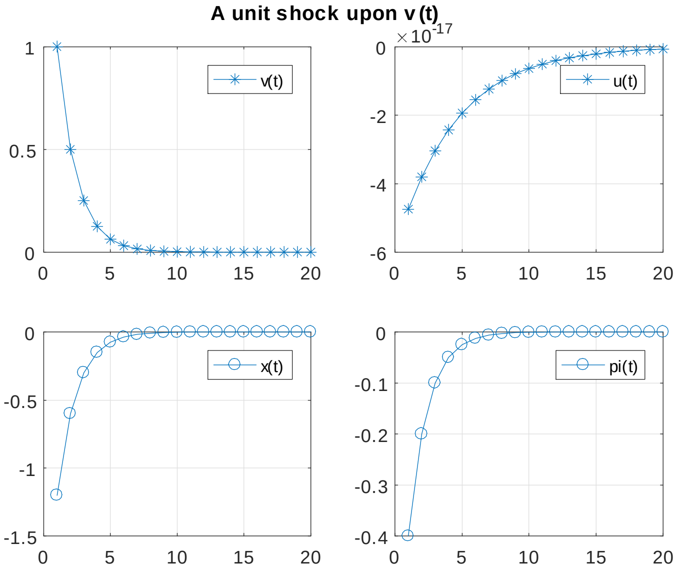
1. The model can be written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{\sigma} \\ 0 & 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} v_{t+1} \\ u_{t+1} \\ E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} =$$

$$\begin{bmatrix} \rho_v & 0 & 0 & 0 \\ 0 & \rho_u & 0 & 0 \\ \frac{1}{\sigma} & 0 & 1 & \frac{1}{\sigma}\delta \\ 0 & -1 & -\kappa & 1 \end{bmatrix} \begin{bmatrix} v_t \\ u_t \\ x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon^v_{t+1} \\ \epsilon^u_{t+1} \end{bmatrix}$$

2. Look at the routine **NKM_Topics_Macro_V2.m** and the IRF that come out of the model

The NKM: IRF from a shock upon v(t)



The NKM: IRF from a shock upon u(t)