ISCTE — INSTITUTO UNIVERSITÁRIO DE LISBOA

Master in Economics

MACROECONOMICS

Problem sets: Models with Rational Expectations

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1 Exercises in the Cagan Model

In 1956, Phillip Cagan published a paper with the title "The Monetary Dynamics of Hyperinflation".[1] The main aim of Cagan was to discuss hyperinflation and the demand for money in such scenarios. He found that the parameters of money demand functions estimated for seven hyperinflations generally satisfied the condition of dynamic stability, which precludes inflation from being self-generating, or displaying period-to-period oscillations. Cagan reached his conclusions by using adaptive expectations. Later on, in 1973 Tom Sargent and Neil Wallace introduced rational expectations into the Cagan's model.[2] The solutions to this model with rational expectations is the essence of this set of exercises.

The basics of the model are three main equations: the demand function for real money balances, the Fisher equation, and the supply of money by a central bank.

If you want to skip the introduction to the model that comes next, and jump right to equation (5) below, you may do so without much loss. Anyway, if you jump, please take into account that p_t is the logarithmic value of the price level P_t and m_t^d is the logarithmic value of the money demand M_t^d. That is $p_t = \ln P_t$ and $m_t^d = \ln M_t^d$.

The demand function for real money holdings. shows this demand to be positively affected by the level of output (Y) and negatively by the cost of money (the nominal interest rate: i). Generically we can write

$$\frac{M_t^d}{P_t} = f(\underset{+}{Y_t}, \underset{-}{i_t})$$

[1] Cagan, Phillip (1956). "The Monetary Dynamics of Hyperinflation". In Friedman, Milton (ed.). Studies in the Quantity Theory of Money. Chicago: University of Chicago Press.

[2] Sargent, Thomas and Wallace, Neil (1973). "The Stability of Models of Money and Growth with Perfect Foresight,"Econometrica, vol. 41(6), pages 1043-48.

Assume that in periods of hyperinflation output is not very important as a determinant of prices, so we can drop it from our investigation. The previous equation can be expressed in log-linear terms as

$$m_t^d - p_t = g(\underset{-}{i_t})) \tag{1}$$

where $m_t^d = \ln M_t^d$ and $p_t = \ln P_t$.

The Fisher equation. This equation reflects the identity that arises in the growing process of a financial investment in nominal and in real terms. It comes out like this

$$1 + i_t = (1 + r_t)(1 + \pi_t)$$

where r_t is the real interest rate and π_t is the inflation rate. If we apply logarithms to the Fisher equation we will get

$$\ln(1 + i_t) = \ln(1 + r_t) + \ln(1 + \pi_t)$$

As the rates under consideration are close to zero, the previous result can be expressed as

$$i_t \approx r_t + \pi_t \tag{2}$$

Now, as far as π_t is concerned, two main points should be highlighted. Firstly, we do not know with certainty the level of inflation during period t, because inflation is the the rate of change of prices during the whole t period (from the beginning of t up to the end of t). So we have to work with expectations about the evolution of inflation (next period's expected price level minus this year price level). Secondly, you know by now very well that the inflation rate is approximately equal to the log difference of prices at $t + 1$ and at t. With all this information we can write

$$\begin{aligned} \pi_t &\approx E_t[\ln P_{t+1}] - \ln P_t \\ &\approx E_t p_{t+1} - p_t \end{aligned}$$

and inserting this back into eq. (2), we will obtain

$$i_t \approx r_t + E_t p_{t+1} - p_t \tag{3}$$

We can now proceed by plugging eq. (3) into eq. (1). The final result is given by

$$m_t^d - p_t = -a r_t - \beta(E_t p_{t+1} - p_t) \tag{4}$$

where a, β are positive parameters. Solving for p_t, we will get

$$p_t = \frac{a}{1+\beta} r_t + \frac{a}{1+\beta} E_t p_{t+1} + \frac{1}{1+\beta} m_t^d. \tag{5}$$

The money supply function. Assume that there are a large number of possible actions by the central bank in terms of the level of money supply it decides to offer. The first option is the money supply to be constant ϕ (remember, in logarithmic terms, we have $m_t^s = \ln\ M_t^s$)

$$m_t^s = \phi. \tag{Supply: Constant}$$

The second possibility, is to have the money supply following a deterministic process like

$$m_t^s = \phi + \theta m_{t-1}^s \quad , \qquad |\theta| < 1 \qquad \text{(Supply: Predetermined)}$$

The third is the money supply following an autoregressive process of order 1, like

$$m_t^s = \phi + \theta m_{t-1}^s + \epsilon_t \quad , \qquad |\theta| < 1 \qquad \text{(Supply: AR1)}$$

where ϵ_t is a set of white noise disturbances.

The fourth possibility is to have the central bank deciding that money will grow at an average rate of 3%. This can be expressed by the following equation

$$m_t^s = \phi + \theta m_{t-1}^s + 0.03 \times t + \epsilon_t \quad , \qquad |\theta| < 1 \qquad \text{(Supply: Stochastic growth rate)}$$

1.1 Questions

Question 1. Explain the economic intuition behind the negative impact of expected inflation upon the real demand for money, as we can observe in eq. (4). *(Just testing some basic economic intuition)*

Question 2. Using all information up to eq. (5) (and only up to this equation), solve for the level of p_t assuming that agents formulate expectations according to the hypothesis of rational expectations. *(Just apply the basic technique learned in classes)*

Question 3. What forces affect the long term equilibrium value of p_t and its stability. Explain. *(Pay attention to small details when answering questions. Look at the problem with care before answering)*

Question 4. Obtain an analytical solution to the dynamics of p_t, in the scenario

$$m_t^s = \phi.$$

Assuming the following set of parameters

$$\phi = 5, \quad r_t = 0.03, \quad a = 6, \quad \beta = 9$$

write down a Matlab routine (or adapt an existing one from our previous classes) that is capable of displaying the time series of p_t for 100 periods. *(Now, you can get a complete answer about the evolution of p_t over time. Prices will depend crucially on the type of dynamics of the money supply (m_t^s). The only thing you have to do is to assume that the money market has to be in equilibrium (so $m_t^s \equiv m_t^d$) and, then, insert the money supply function back into eq. (5). Solve analytically and then simulate it in Matlab.)*

Question 5. Obtain an analytical solution to the dynamics of p_t, in the scenario

$$m_t^s = \phi + \theta m_{t-1}^s \quad , \qquad |\theta| < 1$$

Assuming the following set of parameters

$$\phi = 5, \quad \theta = 0.5, \quad r_t = 0.03, \quad a = 6, \quad \beta = 9,$$

write down a Matlab routine that is capable of displaying the time series of p_t for 100 periods. *(The same as in question 4, but now with a different money supply function)*

Question 6. Obtain an analytical solution to the dynamics of p_t, in the scenario

$$m_t^s = \phi + \theta m_{t-1}^s + \epsilon_t \quad , \quad |\theta| < 1$$

Assuming the following set of parameters

$$\phi = 5, \quad \theta = 0.5, \quad r_t = 0.03, \quad a = 6, \quad \beta = 9,$$

write down a Matlab routine that is capable of displaying the time series of p_t for 100 periods. *(The same as in question 5, but now with a different money supply function. Notice that now you have noise in the supply of money (ϵ_t). You can answer by assuming that we have white noise here ϵ_t˜$N(0,1)$).*

Question 7. *More difficult. You can leave it for later study. Not subject to evaluation in the mid term test.* Obtain an analytical solution to the dynamics of p_t, in the scenario

$$m_t^s = \phi + \theta m_{t-1}^s + 0.03 \times t + \epsilon_t \quad , \quad |\theta| < 1.$$

Can you really obtain an analytical solution to this problem? Notice that the expected value of m_t^s is not constant over time. So $E_t m_{t+i}^s$ increases over time at the value of 0.03.

Hint. Check that $E_t m_{t+i}^s = E_t m_t^s + (0.03 \times i)$.

Assuming the following set of parameters

$$\phi = 5, \quad \theta = 0.5, \quad r_t = 0.03, \quad a = 6, \quad \beta = 9,$$

write down a Matlab routine that is capable of displaying the time series of p_t for 100 periods.

2 An exercise on monetary policy

Assume a type of model that is usually known as the New Keynesian Model (do not worry about its name and where it comes from). This model has three main functions as below: an Aggregate Supply function (AS), the demand side function (IS), and a simple rule for interest rate policy set by the Central Bank. The symbols are as follows: π_t is the inflation rate, x_t is the output gap, i_t is the nominal short term interest rate to be fixed by the central bank.

$$\begin{aligned}
\pi_t &= \beta E_t \pi_{t+1} + \lambda x_t && \text{(Demand function (IS))} \\
x_t &= E_t x_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1}\right) && \text{(AS function)} \\
i_t &= \theta \pi_t && \text{(Central Bank reaction function)}
\end{aligned}$$

where (β, λ, σ) are parameters.

1. Rewrite the model in matricial form, in order to study its stability.
2. For the following set of parameter values $\beta = 0.8$, $\lambda = 0.6$, $\sigma = 2$, study the stability of this model in two different scenarios:

$$\begin{array}{rcl} \text{Scenario A} & : & \theta = 0.5 \\ \text{Scenario B :} & & \theta = 1.1 \end{array}$$

3. What would you conclude about the main message of the New Keynesian Model as far as the fight against inflation is concerned? Explain in the context of the two different values about parameter θ.

Hint: There is an exercise in the set about "Rational Expectations"that has a structure very very similar to this one here. The solutions should also be very similar.

One key point is about the number of eigenvalues that lie outside the unit circle. Remember slide 44 in "Rational Expectations":

If the number of eigenvalues (λ_i) of $A^{-1} \times B$ that lie outside the unit circle ($|\lambda_i| > 1$)

- is equal to the number of forward-looking variables, there exists a unique and stable solution
- is larger than the number of forward-looking variables there is no stable solution
- is lower than the number of forward-looking variables there is an infinity of solutions

3 Exercises on simple dynamic systems (predetermined)

Usually, by now you would not need to waste time on this type of problems. However, if you still feel a little bit uneasy and show some difficulty in solving the previous problems, probably it is a good idea to start by solving these two problems first.

Exercise 1. Consider the following dynamic system of dimension 2

$$\begin{aligned} x_{t+1} &= 3x_t - y_t \\ y_{t+1} &= -2 + 5y_t \end{aligned}$$

(This is a very basic exercise: just for practising)

1. Determine the long term equilibrium of this system.
2. Is its equilibrium unique, or are there multiple equilibria.
3. Using the eigenvalues of the characteristic matrix of this system, conclude about its stability.

ISCTE–IUL, Models with Rational Expectations: Questions, Macroeconomics, Master in Economics, 16/10/2013.

Exercise 2. Consider a dynamic process characterized by the following system

$$
\begin{aligned}
x_{t+1} &= -3x_t + 0.5y_t \\
y_{t+1} &= -2 + 5y_t + \varepsilon_t
\end{aligned}
$$

where ε_t is a IID random variable, with mean equal to zero and constant variance.

1. Determine the long term equilibrium of this system (corresponding to the deterministic component of the system).
2. Characterize this equilibrium in terms of unicity.
3. Using the eigenvalues of the characteristic matrix, conclude about the stability of the system.