Solutions
Rational Expectations
2 Problems discussed in classes

Master in Economics

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Problem 1

Consider that a certain economic process can be represented by the following equation

$$y_t = \alpha + \beta E_t y_{t+1} + u_t$$

where (α, β) are parameters, and u_t is an exogenous variable. $E_t y_{t+1}$ represent the expectations formulated over y_{t+1} with the available information at t.

1. Obtain a solution with rational expectations for the process y_t, assuming that $|\beta| < 1$.

2. What is the importance of parameters (α, β) for the study of the stability of this process? Explain.

3. Do you consider of any particular relevance such kind processes as described by the equation above? Explain.

4. Assume now that u_t corresponds to an Autorregressive process of order 1, (AR1), given by

$$u_t = \psi + \rho u_{t-1} + \varepsilon_t \quad , \quad |\rho| < 1$$

   where ψ is a constant. The term ε_t represents a series of white noise shocks (a random term with identically and independently distributed observations, having mean equal to zero and a constant variance): $\varepsilon_t \tilde{\ } iid(0, \delta^2)$.

5. Obtain a solution with rational expectations for the process y_t, with this additional information.

6. Obtain the same as in the previous question but now assuming that you want to obtain information about short term movements (and not about the long term equilibrium). In order to obtain this result use the following information (for simplicity assume that $\psi = 0$ in this case)
$$E_t u_{t+i} = \rho^i u_t$$

Solution P1.

Sorry, handwritten solutions. See next pages.

1

$$y_t = a + \beta E_t y_{t+1} + u_t$$

$$E_t y_{t+1} = a + \beta E_t y_{t+2} + E_t u_{t+1}$$

$$= a + \beta\left[a + \beta E_t y_{t+2} + E_t u_{t+1}\right] + u_t$$

$$= a + \beta a + \beta^2 E_t y_{t+2} + \beta E_t u_{t+1} + u_t$$

$$E_t y_{t+2} = a + \beta E_t y_{t+3} + E_t u_{t+2}$$

$$= a + \beta a + \beta^2\left[a + \beta E_t y_{t+3} + E_t u_{t+2}\right] + \beta E_t u_{t+1} + u_t$$

$$= a + \beta a + \beta^2 a + \beta^3 E_t y_{t+3} + \beta^2 E_t u_{t+2} + \beta E_t u_{t+1} + u_t$$

$$= \sum_{i=0}^{n-1} \beta^i a + \beta^3 E_t y_{t+3} + \sum_{i=0}^{n-1} \beta^i E_t u_{t+i}$$

$$y_t = \sum_{i=0}^{n-1} \beta^i a + \sum_{i=0}^{n-1} \beta^i E_t u_{t+i}$$

$$= \frac{a}{1-\beta} + \sum_{i=0}^{n-1} \beta^i E_t u_{t+i}$$

2. Stability depends upon β, not upon α.

3. Yes, very relevant

5.

$$u_t = \psi + \rho u_{t-1} + \varepsilon_t \quad , \quad |\rho| < 1$$

long term equilibrium

$$E_t u_{t+1} = \psi + \rho E_t u_t + \underbrace{E_t \varepsilon_{t+1}}_{0}$$

In the long term equilibrium

$$E_t u_{t+1} = E_t u_t = E_t \bar{u}$$

Then

$$E_t \bar{u} = \frac{\psi}{1-\rho}$$

and $E_t u_{t+1} = \dfrac{\psi}{1-\rho}$; $i = 0, 1, \dots$

short term dynamics

$$E_t u_{t+1} = \psi + \rho E_t u_t + \underbrace{E_t \varepsilon_{t+1}}_{0}$$

$$E_t u_{t+2} = \psi + \rho \left[\psi + \rho E_t u_t \right]$$

$$= \psi + \rho \psi + \rho^2 E_t u_t$$

if $\psi = 0$, then this is

$$E_t u_{t+2} = \rho^2 E_t u_t$$

Generalizing

$$E_t u_{t+i} = \rho^i E_t u_t$$

then

$$y_t = \frac{\alpha}{1-\beta} + \frac{\left(\psi/_{1-\rho}\right)}{1-\beta} = \frac{\alpha}{1-\beta} + \frac{\psi}{(1-\beta)(1-\rho)}$$

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$$y_t = \frac{\alpha}{1-\beta} + \sum_{i=0}^{n-1} \beta^i E_t u_{t+i}$$

$$= \frac{\alpha}{1-\beta} + \sum_{i=0}^{n-1} \beta^i \rho^i \underbrace{E_t u_t}_{u_t}$$

$$= \frac{\alpha}{1-\beta} + \sum_{i=0}^{n-1} (\beta\rho)^i u_t = \frac{\alpha}{1-\beta} + \frac{x_t}{1-\beta\rho}.$$

u(t), not x(t)

Problem 2

Consider that a certain economic process can be represented by the following system

$$\begin{aligned}
\pi_t &= \beta E_t \pi_{t+1} + \lambda y_t \\
y_t &= E_t y_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1}\right) \\
i_t &= \bar{\imath}
\end{aligned}$$

where (β, λ, σ) are parameters, i_t is an exogenous variable (or control variable), and $E_t(\cdot)_{t+1}$ represents expectations over $(\cdot)_{t+1}$ with the available information at t.

1. Rewrite this system in matricial form, in order to study its stability.
2. For the following set of parameters, conclude about the kind of stability we have in this system.

$$\beta = 0.8 \ , \quad \lambda = 0.6 \ , \quad \sigma = 2$$

Solution P2.

1. Notice that the system can be written as

$$\begin{aligned}
\beta E_t \pi_{t+1} + 0 E_t y_{t+1} &= \pi_t - \lambda y_t + 0 i_t \\
\frac{1}{\sigma} E_t \pi_{t+1} + 1 E_t y_{t+1} &= 0 \pi_t + 1 y_t + \frac{1}{\sigma} i_t
\end{aligned}$$

In matricial form it looks like this

$$\begin{aligned}
\begin{bmatrix} \beta & 0 \\ \frac{1}{\sigma} & 1 \end{bmatrix} \begin{bmatrix} E_t \pi_{t+1} \\ E_t y_{t+1} \end{bmatrix} &= \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ \frac{1}{\sigma} \end{bmatrix} i_t
\end{aligned}$$

that is

$$\mathbf{A} \cdot \mathbf{E}_t \mathbf{z}_{t+1} = \mathbf{B} \cdot \mathbf{z}_t + \mathbf{C} \cdot \mathbf{v}_t. \tag{1}$$

Now, it is necessary to move matrix $\mathbf{A}$ to the right hand side of eq.(1). To do so we have to use the inverse matrix of $\mathbf{A}$ (that is, $\mathbf{A}^{-1}$), such that we can write

$$\mathbf{E}_t \mathbf{z}_{t+1} = \mathbf{A}^{-1}\mathbf{B} \cdot \mathbf{z}_t + \mathbf{A}^{-1}\mathbf{C} \cdot \mathbf{v}_t$$

Using the small **Note** at the end of this solution (which explains how to calculate the inverse of a 2×2 matrix), you can check that $\mathbf{A}^{-1}$ will be given by

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{\beta} & 0 \\ \\ -\frac{1}{\beta\sigma} & 1 \end{bmatrix}$$

Therefore, the product of $\mathbf{A}^{-1}\mathbf{B}$ is given by

$$\mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} \frac{1}{\beta} & -\frac{\lambda}{\beta} \\ \\ -\frac{1}{\beta\sigma} & \frac{\lambda+\beta\sigma}{\beta\sigma} \end{bmatrix}$$

while

$$\mathbf{A}^{-1}\mathbf{C} = \begin{bmatrix} 0 \\ \\ \frac{1}{\sigma} \end{bmatrix}$$

(notice that, in fact, you do not need this last result for the study of this system's stability, because this stability only depends upon $\mathbf{A}^{-1}\mathbf{B}$).

The system can now be written as

$$\begin{bmatrix} E_t\pi_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} & -\frac{\lambda}{\beta} \\ \\ -\frac{1}{\beta\sigma} & \frac{\lambda+\beta\sigma}{\beta\sigma} \end{bmatrix} \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\sigma} \end{bmatrix} i_t$$

2. With the following parameter values

$$\beta = 0.8 \ , \quad \lambda = 0.6 \ , \quad \sigma = 2$$

the $\mathbf{A}^{-1}\mathbf{B}$ matrix will have the following elements

$$\mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 1.25 & -0.75 \\ \\ -0.625 & 1.375 \end{bmatrix}$$

In order to calculate the two eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$ we should write

$$\left[\mathbf{A}^{-1}\mathbf{B} - \lambda\mathbf{I}_2\right] = \begin{bmatrix} 1.25 - \lambda & -0.75 \\ \\ -0.625 & 1.375 - \lambda \end{bmatrix}$$

where $\mathbf{I}_2$ is an identity matrix of order 2. Obtaining the determinant of $(\mathbf{A} - \lambda\mathbf{I}_2)$ and making it equal to zero

$$\det\left[(1.25 - \lambda)(1.375 - \lambda) - (-0.75 \times -0.625)\right] = 0$$

we obtain

$$\begin{aligned} 1.25 - 1.25\lambda - 1.375\lambda + \lambda^2 &= 0 \\ &\text{or} \\ \lambda^2 - 2.625\lambda + 1.25 &= 0 \end{aligned}$$

$\{[\lambda = 2.0]\,,\ \ [\lambda = 0.625\,]\}$

Notice that this is a quadratic equation. The solutions to the equation are called the roots of the equation and are given by

$$\begin{aligned} \lambda_1, \lambda_2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2.625) \pm \sqrt{(-2.625)^2 - (4 \times 1 \times 1.25)}}{2 \times 1} \end{aligned}$$

Therefore the results are: $\{[\lambda_1 = 2.0]\,, [\lambda_2 = 0.625\,]\}$

As in the current case we have 2 variables "forward looking" and 0 variables pre-determined, in order to have a unique and stable equilibrium, we need two eigenvalues larger than 1 in modulus: $|\lambda_1, \lambda_2| > 1$,and, obviously, 0 eigenvalues lower than 1 (because we have 0 pre-determined variables). However, that is not the case, $|\lambda_1| > 1, |\lambda_2| < 1$. Therefore, the system is indeterminate in this case. If we iterate forward, one of the two variables show stable behavior ($|\lambda_1| > 1$), but the other displays explosive behavior ($|\lambda_2| < 1$). The only way to avoid this explosive behavior is to iterate this variable backwards, but then we end up having an infinite number of equilibria for this variable. This is the meaning of indeterminacy, or sunspots, or animal spirits. Mere beliefs, or forecasting errors, (also called sun spots or non-fundamental economic factors), will drive the dynamics of the economy.

Nota P2: Calculating the inverse of a (2×2) matrix, in the question 1 of Problem 2.

Having the following matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The inverse of $\mathbf{A}$ is calculated as follows

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \end{aligned}$$