S. Shi (2009). Lecture Notes for a course in Macroeconomics, University of Toronto.

Chapter 2
TWO-PERIOD INTERTEMPORAL DECISIONS

The decisions on consumption and savings are at the heart of modern macroeconomics. This decision is about the trade-off between current consumption and future consumption. In this section we describe the necessary techniques for solving such an intertemporal problem.

2.1 An Analogy in Atemporal Choices

To begin, we consider a more familiar decision problem analyzed in intermediate microeconomics – the consumption decision between two goods, say, apples and oranges, at the same date. This is an atemporal decision problem. The standard approach to this problem begins with three elements:

(i) The consumer is assumed to have a preference ordering over these two goods, represented by a utility function $u(c_A, c_O)$, where c_A and c_O are the amount of consumption of apples and oranges, respectively. For illustration, let us use the Cobb-Douglas form of the utility function, $u(c_A, c_O) = (c_A)^\alpha (c_O)^{1-\alpha}$, where $\alpha \in (0, 1)$.

(ii) The consumer takes the relative price between these two goods as given. The relative price of oranges to apples is denoted p.

(iii) The consumer's budget for these two goods is fixed. Let this budget be y, expressed in terms of apples.

With these elements, the consumer's consumption choices, (c_A, c_O), are the solution to the following maximization problem:

$$(P1) \quad \max_{(c_A, c_O)} (c_A)^\alpha (c_O)^{1-\alpha} \tag{2.1}$$

subject to the budget constraint:

$$c_A + pc_O \leq y. \tag{2.2}$$

In intermediate microeconomic textbooks, the solution to this problem is illustrated in a two-dimensional diagram like Figure 2.1. On the horizontal axis is the level of consumption of apples and on the vertical axis is the level of consumption of oranges. The straight, negatively-sloped line is the budget line $c_A + pc_O = y$. The shaded region is the feasibility region. Any bundle of the two goods in this region can be afforded by the consumer under the budget y and any bundle outside the region is not feasible. The curves that are convex to the origin are the indifference curves. Figure 2.1 exhibits three such curves, which correspond to fixed utility levels u_1, u_2, and u_3, respectively. An indifference curve, $u(c_A, c_O) = u_i$, describes all possible combinations of consumption of the two goods that yield the same utility level u_i, some of which may not be feasible. When the indifference curve is further up northeast, it generates higher utility. For example, $u_3 > u_2 > u_1$.

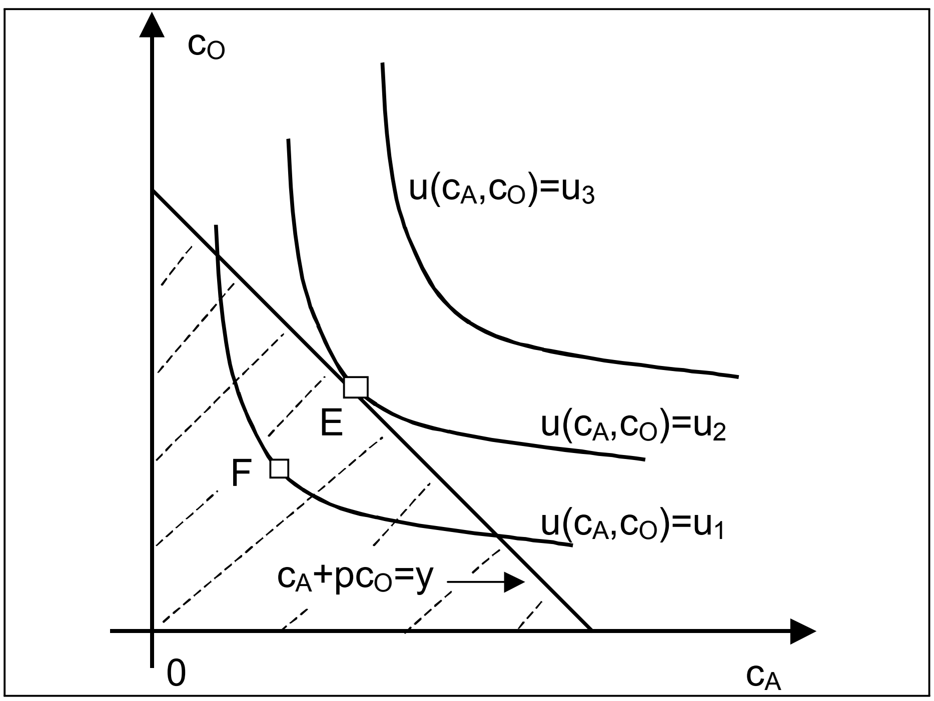

Figure 2.1.

The solution to the above consumption decision problem is point E in Figure 1, where the indifference curve is tangent to the budget line. The optimal solution does not lie on the indifference curve $u(c_A, c_O) = u_3$ because the consumption bundles on this indifference curve are not feasible, i.e., they require a budget higher than y. Similarly, points on the indifference curve $u(c_A, c_O) = u_2$ which are different from E are infeasible. Some consumption bundles on the indifference curve $u(c_A, c_O) = u_1$, such as point F, are feasible, but there are not the optimal choice because they generate lower utility than point E.

The optimal solution represents the best trade-off between consumption of the two commodities under the given relative price. If the agent tries to consume a little bit more apples and fewer oranges than this optimal bundle within the same budget, the consumption point will slide down slightly southeast along the budget line in Figure 2.1. This new consumption bundle will be strictly below the indifference curve $u(c_A, c_O) = u_2$ and so it generates a lower utility level than the optimal bundle E. Similarly, if the consumer tries to consume a bit more oranges and a bit fewer apples within the same budget, the consumption bundle slides up northwest along the budget line, again yielding a lower utility level.

2.2 A Two-Period Model

The intertemporal consumption decision can be analyzed in a way very similar to the above atemporal problem. Rather than choosing between consumption of different goods at the same date, the consumer now chooses between consumption at different dates. To simplify the illustration of this intertemporal problem, let us assume that the consumer lives only for two periods, date 0 and date 1, and that the goods consumed at different dates are physically the same. The consumption level is c_0 at date 0 and c_1 at date 1. We start with three elements that are similar to those in the atemporal problem.

(i) The consumer is assumed to have a preference ordering over consumption at the two dates, represented by a utility function $u(c_0, c_1)$. Since such a utility index involves consumption at different dates, we call it the "intertemporal utility function".

(ii) The consumer takes the gross real interest rate, R, as given.

(iii) The consumer's budget at date 0 is y, expressed in terms of date 0 goods. To simplify, assume that the only source of income for the consumer at date 1 is the income from savings at date 0.

At date 0, the consumer chooses the consumption level, c_0, and the amount of savings, which is denoted s_0. Savings are in the form of consumption goods at date 0 and so it has the same price as consumption goods at date 0. The consumer's budget constraint at date 0 is

$$c_0 + s_0 \leq y. \tag{2.3}$$

At date 1, the consumer chooses the consumption level, c_1. The income in this period is the income from savings at date 0. Since each unit of consumption goods saved at date 0 yields R units of consumption goods at date 1, the consumer's income at date 1 is Rs_0. The budget constraint at date 1 is

$$c_1 \leq Rs_0. \tag{2.4}$$

Example 1 *In the story of Robinson Crusoe, Crusoe found a fixed quantity of corns on the deserted island and faced the decision of how much of the corns to consume. Crusoe's problem is similar to the one described above. The initial endowment of corns serves the role of* y *in the above constraints. If he eats all the corns, he cannot consume any the next year. If he saves some corns and sow them as seeds, he can obtain new corns next year. The seeds are capital, although they have the same physical form as the consumption good.*

The consumer's consumption decisions in the two periods are the solution to the following maximization problem:

$$\max\ u(c_0, c_1) \text{ subject to (2.3) and (2.4).}$$

In contrast to the atemporal decision problem, this intertemporal maximization problem has two budget constraints, one for each period. This difference, however, is superficial. We can easily transform the intertemporal problem into exactly the same form as the atemporal problem. As a reasonable assumption, we can assume that the consumer is happier if his/her consumption is higher. (Later we specify explicitly the assumptions on u.) In this case the consumer will never throw any part of his/her income away at date 1. So, (2.4) holds with equality. Substituting $s_0 = c_1/R$ into (2.3), we obtain:

$$c_0 + \frac{1}{R}c_1 \leq y. \tag{2.5}$$

The intertemporal problem can thus be rewritten as

$$(P2) \quad \max\ u(c_0, c_1) \text{ subject to (2.5).}$$

This problem has the same mathematical form as the atemporal problem $(P1)$. The budget constraint, (2.5), is called the intertemporal budget constraint.

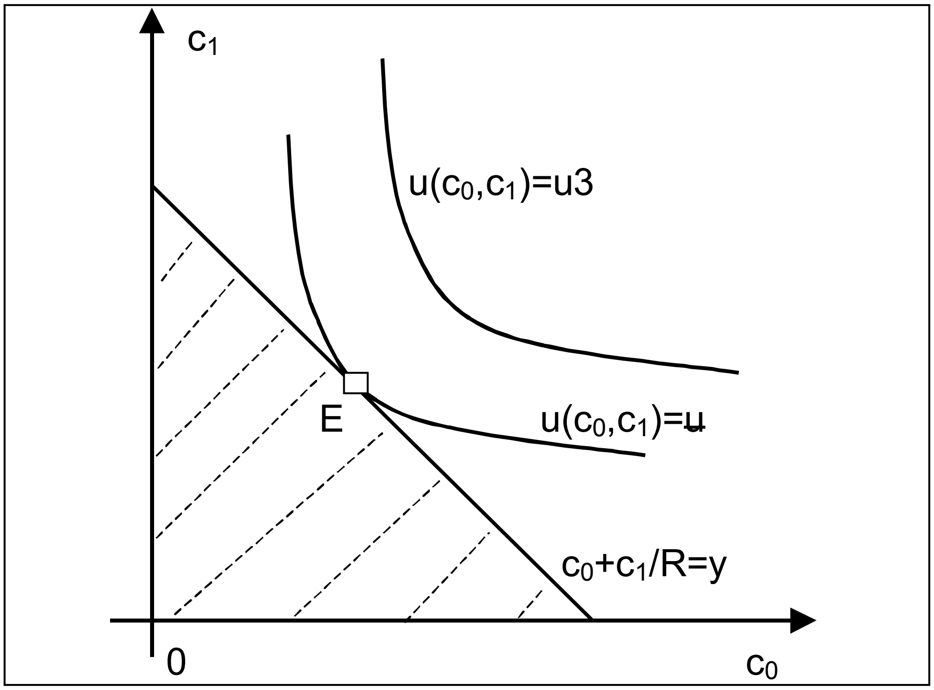

Figure 2.2.

The similarity between the two decision problems allows us to emphasize two aspects of an intertemporal decision problem:

- In the atemporal problem the two goods at the same date are substitutes and the optimal decision is such a bundle at which further substitution between the two goods does not increase utility. In the intertemporal problem the trade-off between current consumption and savings is a trade-off between current consumption and future consumption. That is, the consumer can substitute consumption across periods. We call this substitution the "intertemporal substitution". The optimal intertemporal decision is such an allocation (c_0, c_1) that further substitution between consumption at the two dates does not increase intertemporal utility.

- In the atemporal problem, p is the relative price of oranges to apples at the same date. In the intertemporal problem the relative price of date 1 good to date 0 good is $1/R$. That is, the gross real interest rate is the relative price of date 0 good to date 1 good. This is the intertemporal price. It is not one if the net real interest rate is positive, although the two goods have the same physical form.

We can draw Figure 2.2 to depict the problem ($P2$). The straight, negatively-sloped line is the equality form of the intertemporal budget constraint, the shaded area is the feasibility set, the convex curve is a particular indifference curve, and point E is the solution. The budget line is also called the feasibility frontier.

For the solution to exist and to be unique, as in Figure 2.2, the indifference curve must be decreasing and convex. This requires the utility function to satisfy certain conditions. To elaborate, consider first the example with $u(c_0, c_1) = (c_0)^{\alpha}(c_1)^{1-\alpha}$, where $\alpha \in (0, 1)$. In this example, for any fixed utility level $\underline{u}$ the indifference curve is $(c_0)^{\alpha}(c_1)^{1-\alpha} = \underline{u}$. That is,

$$c_1 = (\bar{u})^{\frac{1}{1-\alpha}}(c_0)^{-\frac{\alpha}{1-\alpha}}. \tag{2.6}$$

This indifference curve expresses c_1 as a decreasing function of c_0, a feature illustrated in Figure 2.2. That is,

$$\left.\frac{dc_1}{dc_0}\right|_{u=\underline{u}} = -\frac{\alpha}{1-\alpha}(\underline{u})^{\frac{1}{1-\alpha}}(c_0)^{-\frac{\alpha}{1-\alpha}-1} < 0,$$

where we use the symbol $(.)|_{u=\underline{u}}$ to emphasize the fact that the indifference curve fixes the utility level at some level $\underline{u}$. The indifference curve is also a convex function, as illustrated in Figure 2.2. That is,

$$\left.\frac{d^2c_1}{dc_0^2}\right|_{u=\underline{u}} = \frac{\alpha}{1-\alpha}\left(\frac{\alpha}{1-\alpha}+1\right)(\underline{u})^{\frac{1}{1-\alpha}}(c_0)^{-\frac{\alpha}{1-\alpha}-2} > 0.$$

For a general utility function $u(c_0, c_1)$, we cannot express c_1 as a function of c_0 explicitly as in (2.6). Nevertheless, we can find conditions under which the indifference curve has the shape in Figure 2.2. First, differentiating the definition of the indifference curve $u(c_0, c_1) = \underline{u}$, we have

$$(u_1 dc_0 + u_2 dc_1)|_{u=\underline{u}} = 0,$$

where u_i is the derivative of $u(c_0, c_1)$ with respect to its ith argument/variable ($i = 1, 2$). From this we can solve for the slope of the indifference curve at the allocation (c_0, c_1) as

$$\left.\frac{dc_1}{dc_0}\right|_{u=\underline{u}} = -\frac{u_1}{u_2}.$$

As a reasonable assumption, we require utility to increase with the amount of consumption, i.e., the more the better. This requires $u_1 > 0$ and $u_2 > 0$. This assumption also ensures that the indifference curve is downward sloping at any allocation (c_0, c_1).

Second, we find the curvature of the indifference curve by totally differentiating the slope of the indifference curve:

$$\left.\frac{d^2 c_1}{dc_0^2}\right|_{u=\underline{u}} = \frac{1}{u_2^2}\left[u_1\left(u_{12} + u_{22}\left.\frac{dc_1}{dc_0}\right|_{u=\underline{u}}\right) - u_2\left(u_{11} + u_{12}\left.\frac{dc_1}{dc_0}\right|_{u=\underline{u}}\right)\right].$$

In this expression, u_{ij} is defined as $u_{ij} \equiv \partial^2 u/(\partial c_i \partial c_j)$ for $i, j = 1, 2$. Substituting the derivative dc_1/dc_0, we have

$$\left.\frac{d^2 c_1}{dc_0^2}\right|_{u=\underline{u}} = -\frac{1}{u_2^3}\left(u_2^2 u_{11} + u_1^2 u_{22} - 2u_1 u_2 u_{12}\right).$$

The indifference curve is convex, as drawn in Figure 2.2, if and only if the above derivative is positive, i.e., if and only if

$$u_2^2 u_{11} + u_1^2 u_{22} - 2u_1 u_2 u_{12} < 0. \tag{2.7}$$

This feature of the utility feature is called quasi-concavity.

Finally, we need the marginal utility to be a decreasing function of consumption. That is, as consumption at a date increases and consumption at the other date remains fixed, the increment in utility diminishes. This requires $u_{11} < 0$ and $u_{22} < 0$.

We can summarize the assumptions as follows, which we maintain throughout our analysis:

Assumption 1 *The utility function $u(c_0, c_1)$ satisfies the following conditions:*
(i) Positive marginal utility: $u_1 > 0$ and $u_2 > 0$.
(ii) Diminishing marginal utility: $u_{11} < 0$ and $u_{22} < 0$.
(iii) Quasi-concavity: $u_2^2 u_{11} + u_1^2 u_{22} - 2u_1 u_2 u_{12} < 0$.

Quasi-concavity corresponds to the convexity of the indifference curves and hence is necessary for the uniqueness of the solution to the maximization problem. It is different from the common concept of concavity when a function has two or more arguments. A function $u(c_1, c_2)$ is concave jointly in (c_0, c_1) if for any two consumption programs $C = (c_0, c_1)$ and $C^* = (c_0^*, c_1^*)$, and any $\alpha \in (0, 1)$, the function u satisfies

$u(C_\alpha) > \alpha u(C^*) + (1-\alpha)u(C)$, where $C_\alpha \equiv \alpha C^* + (1-\alpha)C$. This concavity condition can be rewritten as*

$$u_{11}u_{22} - u_{12}^2 > 0. \tag{2.8}$$

It is easy to verify that concavity implies quasi-concavity. But the reverse is not true, as shown in the following exercises.

Exercise 2.2.1 *The utility function $u(c_0, c_1) = (c_0)^\alpha (c_1)^{1-\alpha}$, where $\alpha \in (0,1)$, satisfies Assumption 1. Show that this utility function is not concave in (c_0, c_1) jointly.*

Exercise 2.2.2 *The function, $u(c_0, c_1) = [\alpha c_0^\rho + (1-\alpha)c_1^\rho]^{1/\rho}$, where $\alpha \in (0,1)$ and $\rho < 1$, is called the utility function with a constant elasticity of substitution between c_0 and c_1 (CES for short). Show that the CES function is quasi-concave but is not concave in (c_0, c_1) jointly.*

A special type of intertemporal utility function is the following:

$$u(c_0, c_1) = U(c_0) + \beta U(c_1), \;\; \beta > 0, \tag{2.9}$$

This intertemporal utility function assumes that the consumer derives utility from consumption in each period separately and that intertemporal utility is a weighted sum of the utility levels in the two periods. This type of intertemporal utility function is called the time-additive utility function. The relative weight of future utility to current utility, β, is called the discount factor. The discount rate is $\frac{1}{\beta} - 1$.

Exercise 2.2.3 *Show that the time-additive utility function satisfies Assumption 1 if and only if $U' > 0$ and $U'' < 0$.*†

*To see why (2.8) is necessary for concavity, let $\Delta_0 = c_0^* - c_0$ and $\Delta_1 = c_1^* - c_1$. Then, $C^* = (c_0 + \Delta_1, c_1 + \Delta)$ and $C_\alpha = (c_0 + \alpha\Delta_0, c_1 + \alpha\Delta_1)$. Consider the function $f(\Delta_0, \Delta_1) \equiv u(C_\alpha) - [\alpha u(C^*) + (1-\alpha)u(C)]$. If u is concave, then $f(\Delta_0, \Delta_1) > 0$ for all (Δ_0, Δ_1), and certainly for sufficiently small (Δ_0, Δ_1). Note that $f(0,0) = 0$ and $f_1(0,0) = f_2(0,0) = 0$. Using Taylor expansion to expand $f(\Delta_0, \Delta_1)$ around $(0,0)$ and ignoring the terms close to zero, we have:

$$f(\Delta_0, \Delta_1) = -\frac{\alpha}{2}(1-\alpha)[u_{11}\Delta_0^2 + 2u_{12}\Delta_0\Delta_1 + u_{22}\Delta_1^2],$$

where the derivatives u_{11}, u_{12} and u_{22} are all evaluated at $(\Delta_0, \Delta_1) = (0,0)$. For $f(\Delta_0, \Delta_1) > 0$ around $(\Delta_0, \Delta_1) = (0,0)$, it is necessary and sufficient that $u_{11}\Delta_0^2 + 2u_{12}\Delta_0\Delta_1 + u_{22}\Delta_1^2 < 0$. This condition is equivalent to that the matrix $\begin{pmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{pmatrix}$ be negative definite, i.e., $u_{11} < 0$, $u_{22} < 0$ and $u_{11}u_{22} - u_{12}^2 > 0$.

†When a function has only one argument, we often use $'$ to denote the first-order derivative and $''$ the second-order derivative of the function with respect to that argument.

2.3 The Lagrangian Method

The diagrammatic approach in the last section is often inadequate for macroeconomic analysis. We need to find the conditions that determine the solution of the maximization problem. To solve ($P2$), there are a few methods. All of them use the following idea:

If (c_0^*, c_1^*) is the optimal intertemporal consumption allocation, then intertemporal utility cannot be increased by any other feasible allocation.

The most common way to implement this idea is the Lagrangian method, which has the following steps:

Step 1. Rewrite the intertemporal budget constraint by moving all terms to one side of the condition and express the condition in the form with "$\geq$". For (2.5) this step generates $y - c_0 - c_1/R \geq 0$.

Step 2. Multiply the left-hand side of the rewritten budget constraint by a multiplier, say λ, and add this term to the objective function of the maximization problem. This creates

$$\mathcal{L} \equiv u(c_0, c_1) + \lambda \left(y - c_0 - \frac{c_1}{R} \right). \tag{2.10}$$

Step 3. Take the derivatives of the above function with respect to c_0 and c_1. For the consumption choices to be optimal, such derivatives are zero except for some special cases discussed below. Setting the derivatives to zero we obtain the first-order conditions of the maximization problem. The solution to these first-order conditions is the solution to ($P2$).

Step 1 is self-explanatory. Step 2 combines the intertemporal budget constraint with the objective function of the maximization problem. The multiplier λ is called the Lagrangian multiplier. In the current case it is also called the shadow price of income at date 0, measured in terms of date-0 utility. This price measures how much date-0 utility a marginal unit of income can bring. It is the "shadow" price because it is not a price typically observed in the market. The function $\mathcal{L}$ in (2.10) is the Lagrangian of the original maximization problem. In Step 3, we take λ as given and maximize $\mathcal{L}$ by choosing c_0 and c_1.

For general forms of the utility function, the choices c_0 and c_1 that maximize $\mathcal{L}$ are given by the following first-order conditions:

$$\frac{\partial L}{\partial c_0} = u_1 - \lambda \leq 0, \quad = 0 \text{ if } c_0 > 0, \tag{2.11}$$

$$\frac{\partial L}{\partial c_1} = u_2 - \lambda/R \leq 0, \quad = 0 \text{ if } c_1 > 0, \tag{2.12}$$

and the requirement:

$$\lambda\,(y - c_0 - c_1/R) = 0. \tag{2.13}$$

Eq. (2.11) states that the net marginal gain in utility from increasing c_0 should be zero if c_0 is positive, and non-positive if c_0 is zero. Eq. (2.12) states a similar condition for c_1. Eq. (2.13) is a more general form of the intertemporal budget constraint. It states that the budget constraint holds with equality if the shadow price of income is positive and that the only case where the budget constraint holds with inequality is when the shadow price of income is zero.

The optimal choice of c_0 (or c_1) can be positive or zero. When the optimal choice of c_0 is positive, (2.11) holds with equality, in which case the solution is an "interior solution". The equality form of (2.11) requires the optimal choice of c_0 to equate the marginal benefit and the marginal cost of c_0. The marginal benefit of c_0 is the marginal utility, u_1. The marginal cost of c_0 is equal to the shadow price of income, λ, because each unit of increase in c_0 entails an equal amount of income. To see why the optimal choice of c_0 must satisfy this condition, suppose counterfactually that $u_1 > \lambda$. Then, if the consumer "pays" λ units of utility to get a marginal unit of income and use it to increase consumption c_0, he will obtain u_1 units of utility from the additional income. The net gain in utility is $(u_1 - \lambda) > 0$. This means that the original consumption level is too low to be optimal. The consumer can keep increasing utility by increasing consumption. As consumption increases, marginal utility of consumption decreases and eventually is equal to the marginal cost of consumption. This is the level at which further increases in consumption do not generate positive net utility. Similarly, if $u_1 < \lambda$, the consumption level is too high and the consumer can increase utility by reducing consumption until net utility $(u_1 - \lambda)$ becomes zero at some optimal level c_0^* (if $c_0^* > 0$).

It is possible that optimal consumption is zero, i.e., the constraint $c_0 \geq 0$ binds, in which case the solution is a "corner solution". This happens when net utility $(u_1 - \lambda)$ is negative even when c_0 is reduced all the way to 0. Since consumption must be non-negative, the choice $c_0 = 0$ is the optimal choice in this case and so the condition (2.11) holds with inequality. The following exercise provides an example.

Exercise 2.3.1 *Assume that* $u(c_0, c_1) = \ln(c_0 + A) + \ln c_1$ *and that* $y < A$. *Show that this utility function satisfies Assumption 1 but the optimal choice of* c_0 *is* 0.

To check that the Lagrangian method indeed yields the solution to the original maximization problem ($P2$), we show that the choices that maximize the Lagrangian are indeed the solutions to the original problem. Let (c_0^*, c_1^*) maximize the Lagrangian. To simplify the illustration, assume $c_0^* > 0$, $c_1^* > 0$ and so (2.11) and (2.12) hold with equality. Then $\lambda = u_0 > 0$. Eliminating λ from (2.11) – (2.13) we have:

$$u_1/u_2 = R, \tag{2.14}$$

$$y - c_0 - c_1/R = 0. \tag{2.15}$$

Suppose that the choices (c_0^{**}, c_1^{**}) solve the original problem ($P2$), they must satisfy (2.14) and (2.15), and so they are equal to (c_0^*, c_1^*). To see this, we solve the original problem directly. Note that the assumptions $u_1, u_2 > 0$ imply that the consumer will not want to waste any income and so the budget constraint holds with equality. That is, (c_0^{**}, c_1^{**}) satisfy (2.15). Using this budget constraint we can write $c_1 = R(y - c_0)$. Substituting this into the utility function, we write the objective function as $u(c_0, R(y - c_0))$. The optimal choice of c_0, c_0^{**}, maximizes this function. The first-order condition is precisely (2.14).

Remark 1 *We have referred to the Lagrangian multiplier as the shadow price of income. This can be made rigorous by showing that $d\mathcal{L}/dy = \lambda$. In fact, if the consumption profile is chosen optimally, then $d\mathcal{L}/dy = \partial\mathcal{L}/\partial y$ $(= \lambda)$. The equality between the total derivative and the partial derivative of the Lagrangian to income is called the "envelope theorem".*

The direct approach to the original maximization problem, described above, seems easier than the Lagrangian method. What is the use of the Lagrangian method, then? The answer is that the Lagrangian method is more general. The Lagrangian method allows for corner solutions to the maximization problem and is applicable to maximization problems that have complicated constraints. In the maximization problem described here, the budget constraint is simple and we can use it to express c_1 as a function of c_0. In many other problems, however, the direct approach is awkward because it may not be possible to use the constraints to explicitly express some variables as functions of other variables. The following exercise gives an example.

Exercise 2.3.2 *Let the consumer's utility function $u(c_0, c_1)$ be strictly increasing in each argument and quasi-concave in the two arguments jointly. Suppose that the consumer must incur some "shopping cost" in addition to the costs of goods in order* *to obtain the consumption bundle. The shopping cost, expressed in terms of period-*0 *goods, is given by a function* $\phi(c_0, c_1)$*, which is strictly increasing and convex in each argument. Incorporate this shopping cost into the consumer's intertemporal budget constraint. Use the Lagrangian method to derive the first-order conditions of this problem. Is the marginal utility of consumption in period* 0 *equal to the marginal utility of income in this case?*

The optimal condition, (2.14), is an important condition. The ratio on the left-hand side, u_1/u_2, is called the marginal rate of substitution between current and future consumption. It measures the household's relative value of current consumption to future consumption, or equivalently, how much the consumer is *willing to pay* in future consumption for one unit of increase in current consumption. The real interest rate acts as the marginal rate of transformation between currency and future consumption. It is the rate of return to savings, measuring the rate of substitution between current and future consumption that is *available in the market.* If the household's consumption profile (c_0, c_1) is optimal, then how much the household is willing to pay to substitute between current and future consumption must be equal to what the market can provide. Thus (2.14) must hold. If the marginal rate of substitution is higher than the real interest rate, the return to savings does not compensate sufficiently the sacrifice of current consumption that the household makes. In this case the household can increase intertemporal utility by reducing savings and increasing current consumption. On the other hand, if the marginal rate of substitution is lower than the real interest rate, the return to savings exceeds the sacrifice of current consumption that the household makes. In this case the household can increase intertemporal utility by increasing savings and reducing current consumption. The consumption profile (c_0, c_1) is optimal only when the household does not have any net gain by increasing or decreasing savings at the margin.

2.4 Determinants of Savings

Once we have obtained the optimal conditions of the maximization problem, we can use them to address economic issues. This requires us to re-organize the optimal conditions. How to re-organize the conditions depends on what issues are to be discussed. Here we discuss the factors that determine optimal savings.

Savings are important for capital formation and economic growth. The simple two-period model reveals three important determinants of savings: the income profile, the rate of return to savings (the real interest rate) and the agent's patience toward future consumption. Let us interpret the consumer in the two-period model as the average household in a large economy and so the savings analyzed here are average savings per household in the economy. To simplify illustrations, we assume that the household's intertemporal is the time-additive function in (2.9).

2.4.1 A Slightly More General Model

We enrich the two-period model by allowing the household to have income at date 1 in addition to the income from savings. For example, the household might obtain labor income at date 1. Let y_1 be such income. Let y_0 now denote the level of income at the beginning of date 0. Then the household's budget constraint at date 1 is

$$c_1 \leq Rs_0 + y_1.$$

The household's budget constraint at date 0 is (2.3), with y being replaced by y_0. The intertemporal constraint is

$$c_0 + \frac{c_1}{R} \leq y_0 + \frac{y_1}{R}. \tag{2.16}$$

The level of savings is $s_0 = y_0 - c_0$. To determine the optimal level of savings, we need to determine the optimal level of consumption at date 0. Thus, we should find the optimal condition for c_0 and replace other variables in such a condition as functions of c_0 whenever it is possible.

The optimal c_0 is given by (2.14). Since the intertemporal budget constraint (2.16) binds (i.e., holds with equality), $c_1 = y_1 + R(y_0 - c_0)$. Substituting this for c_1 in (2.14) and using the time-additive utility function, we obtain the condition for the optimal level of c_0:

$$\frac{U'(c_0)}{U'(y_1 + R(y_0 - c_0))} = R\beta. \tag{2.17}$$

2.4.2 Income Profile

By the income profile we refer specifically to the household's income derived from human capital such as the labor income, not counting income derived from savings. In the current model, the income profile is (y_0, y_1). To illustrate how the income profile affects savings, assume that the gross real interest rate is $R = 1$ and that the household does not discount future utility (i.e., $\beta = 1$). In this case the condition for the optimal level of c_0, (2.17), becomes:

$$U'(c_0) = U'(y_1 + y_0 - c_0).$$

Since the marginal utility of consumption, U', is a decreasing function, we must have $c_0 = y_1 + y_0 - c_0$ $(= c_1)$. That is, the household consumes the same amount in the two periods. We call this motive to achieve similar consumption levels in different periods the "consumption-smoothing motive". Savings are a means to smooth consumption over time.

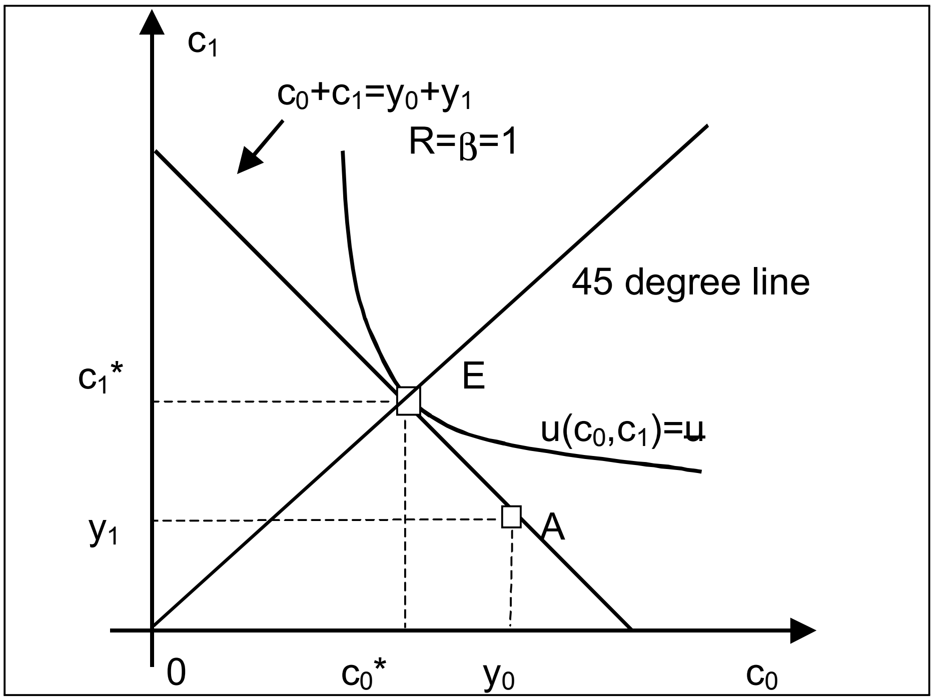

Figure 2.3. An uneven income profile
(The case with $R = \beta = 1, y_0 > y_1$).

In the current case we can explicitly solve for the optimal level of savings. Since $c_0 = y_1 + y_0 - c_0$, then $c_0 = (y_1 + y_0)/2$. The optimal level of savings is

$$s_0 = y_0 - c_0 = \frac{y_0 - y_1}{2}.$$

If the household has a flat income profile (and if $R = \beta = 1$), the optimal level of savings is 0; if the household has a rising income profile (i.e., if $y_1 > y_0$), the optimal level of saving is negative; if the household has a declining income profile (i.e., if $y_1 < y_0$), the optimal level of savings is positive. A negative level of savings means that the household borrows at date 0.

In Figure 2.3 we illustrate the case in which the household has a declining income profile. The optimal consumption profile (c_0^*, c_1^*) is at point E where the indifference curve is tangent to the intertemporal budget line. In the current case, the intertemporal budget line is $c_0 + c_1 = y_0 + y_1$, since $R = 1$. Because the optimal consumption profile satisfies $c_0^* = c_1^*$ in the current case, point E lies on the 45^0 line.

The endowment point is point A. The endowment point lies on the intertemporal budget line because it is always feasible to consume the entire income in each period. The endowment point lies southeast of the optimal consumption point because $y_0 > y_1$. Since $y_0 > c_0^*$, the level of savings at date 0 is positive.

Exercise 2.4.1 *Explain why savings often increase with age when the agent is young and then decline with age when the agent retires.*

Exercise 2.4.2 *Explain whether each of the following statements is true, false, or uncertain:*
(i) The low saving rate in the US must indicate that growth in income per capita has slowed down in the US.
(ii) If households anticipate that the income tax will rise in the future, then they will increase savings.

2.4.3 Real Interest Rate

The second determinant of optimal savings is the rate of return to savings, i.e., the real interest rate. To isolate the role of the real interest rate, we assume that the household does not discount future utility ($\beta = 1$) and, for the moment, that the household has a smooth income profile ($y_1 = y_0 = y$). In the case, the optimal level of savings is 0 if the real interest rate is $R = 1$. Assume, instead, that $R > 1$.

With $\beta = 1$ and $y_1 = y_0 = y$, the condition for the optimal level of consumption c_0, (2.17), becomes

$$U'(c_0) = R \times U'(y + R(y - c_0)).$$

Unlike the previous case, the optimal consumption levels are different in the two periods. Given the smooth income profile and no time discounting, a higher real interest rate entices the household to save more.

Let us illustrate this positive effect of the real interest rate on savings in Figure 2.4. Point E is the optimal consumption choice when $R = 1$ (and $y_1 = y_0$, $\beta = 1$). At this point, the optimal level of savings is zero and so the optimal consumption level in each period is equal to the income level in that period. When the real interest rate increases from 1 to a number greater than 1, the intertemporal budget line rotates counter-clockwise around the point of the income profile, i.e., around point E. The new intertemporal budget line is steeper than the original one and the new optimal consumption profile is given by point E'. Since point E' is above the 45^0 line, the household consumes more at date 1 than date 0, although the household's income profile is completely flat. The level of savings increases from 0 to a positive level.

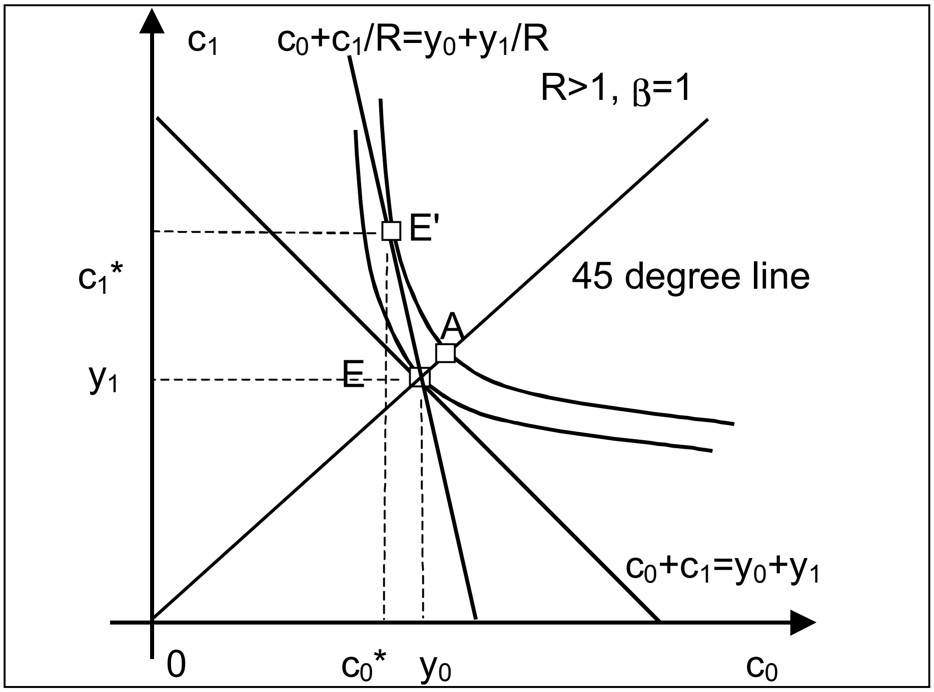

Figure 2.4. An increase in the real interest rate
(The case with $\beta = 1, y_0 = y_1 = y$)

To explain why the household saves a positive amount in the case depicted in Figure 2.4, suppose counter-factually that the household chooses zero savings. Since the household's income is the same in the two periods, this means $c_0 = c_1 = y$. The household's utility is $U(y)$ in each period and, since the household does not discount future utility in the current case, intertemporal utility is $2U(y)$. Now suppose that the household tries to save a slightly positive amount $\delta > 0$ and reduce consumption at date 0 by the same amount. The budget constraint at date 0 holds with equality. With the savings, the household will be able to consume $c_1 = y + R\delta$ in period 1. This new consumption profile gives the household intertemporal utility $U(y-\delta)+U(y+R\delta)$. Since δ is very small, the difference between this utility level and the one generated by a flat consumption profile is:

$$\begin{aligned}&[U(y-\delta)+U(y+R\delta)]-2U(y)\\&=[U(y-\delta)-U(y)]+[U(y+R\delta)-U(y)]\\&\approx -U'(y)\delta+U'(y)R\delta=(R-1)U'(y)\delta>0.\end{aligned}$$

By saving a small amount δ and hence consuming δ less at date 0, the household experiences a reduction in utility at date 0 of the magnitude $U'(y)\delta$. However, the savings enable the household to increase consumption by $R\delta$ at date 1, which increases utility at date 1 by an amount $U'(y)R\delta$. With no discounting of future utility, $R > 1$ implies that the gain outweighs the loss from saving a small amount.

Exercise 2.4.3 *Assume* $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$*, where* $\sigma > 0$*. With* $\beta = 1$ *and* $y_1 = y_0 = y$*, show that the optimal level of savings in the two-period economy is*

$$s_0^* = \frac{R^{1/\sigma} - 1}{R^{1/\sigma} + R} y.$$

Furthermore, show that $s_0^* > 0$ *if and only if* $R > 1$*.*

The situation depicted in Figure 2.4 illustrates an important effect of the interest rate on savings — the intertemporal substitution effect. This effect arises from the fact that an increase in the real interest rate effectively makes consumption cheaper at date 1 than at date 0. (Recall that the real interest rate is the relative price of date 0 goods to date 1 goods.) As the relative price of date-0 goods rises, the household substitutes away from consumption of date-0 goods and into consumption of date-1 goods. Therefore, the substitution effect of an increase in the real interest rate always increases savings.

An increase in the interest rate also creates an income effect on savings, which Figure 2.4 suppresses. In Figure 2.4, the household has zero savings before the increase in the interest rate, and so the increase in the interest rate does not affect the household's future income. If, instead, the household has non-zero savings, then an increase in the interest rate will affect the household's interest income, Rs_0, thus changing future income. Suppose, for example, that the household has positive savings. An increase in the interest rate increases the household's interest income from savings, and hence increases future income relative to current income. Anticipating this rising income profile, the household reduces savings in order to smooth consumption. On the other hand, if the household has negative savings, an increase in the interest rate increases the household's interest payment in date 1. In this case, future income falls relative to current income, and the household increases savings in order to smooth consumption.

Therefore, the income effect and the intertemporal substitution effect of the interest rate on savings work in the same direction when the households has negative savings but opposite directions when the household has positive savings. When they work in opposite directions, savings increase with the interest rate if and only if the intertemporal substitution effect is stronger. The following example shows that for a particular utility function, the intertemporal substitution effect dominates.

Example 2 *Assume* $U(c) = \ln c$, $\beta = 1$ *and* $y_0 > y_1$. *Then, the household's optimal decisions yield*

$$c_0 = \frac{c_1}{R} = \frac{1}{2}\left(y_0 + \frac{y_1}{R}\right).$$

The level of savings is $s_0 = y_0 - c_0 = \frac{1}{2}\left(y_0 - \frac{y_1}{R}\right)$. *Clearly,* $s_0 > 0$ *for any* $y_0 > y_1$, *and so an increase in* R *generates an income effect that tends to reduce savings. This income effect is dominated by the intertemporal substitution effect, as* $ds_0/dR > 0$.

A measure of the strength of intertemporal substitution is the elasticity of intertemporal substitution. Because R is the relative price between goods at different dates, we can define the intertemporal elasticity as

$$-\frac{d\ln(c_0/c_1)}{d\ln(R)}.$$

In the above example, $c_0/c_1 = 1/R$, and so the elasticity of intertemporal substitution is 1. Another way to calculate the elasticity is $-\frac{d\ln(c_0/c_1)}{d(u_1/u_2)}$, which comes from substituting the first-order condition $R = u_1/u_2$ in the definition of the intertemporal elasticity. For the time-additive utility function, with $U(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, we can calculate the elasticity of intertemporal substitution as $1/\sigma$. The logarithmic utility function is a special case of this, with $\sigma = 1$.

The following exercise shows how the elasticity of intertemporal substitution determines the balance between the two effects of the interest rate on savings.

Exercise 2.4.4 *When the utility function is* $U(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, *where* $\sigma > 0$, *the elasticity of intertemporal substitution is* $1/\sigma$. *Assume* $\beta = 1$, $R \geq 1$, $y_0 > 0$ *and* $y_1 > 0$. *Prove the following results.*

(i) The optimal level of savings in the two-period economy is

$$s_0^* = \frac{y_0 R^{1/\sigma} - y_1}{R^{1/\sigma} + R}.$$

(ii) $s_0^* \leq 0$ *always implies* $ds_0^*/dR > 0$.

(iii) If $R = 1$ *and* $y_0 < y_1$, *then* $ds_0^*/dR > 0$ *if and only if* $\sigma < (y_0 + y_1)/(y_0 - y_1)$.

2.4.4 Impatience toward Future

The third determinant of savings is the degree of impatience which the household has toward future consumption and utility. Intuitively, the more impatient the household is toward future, the less the savings are. When the intertemporal utility function has the additive form, (2.9), it is easy to see that the discount factor, β, measures the degree of patience and so the discount rate $(1/\beta - 1)$ measures the degree of impatience. We sometimes call the latter the "rate of time preference".

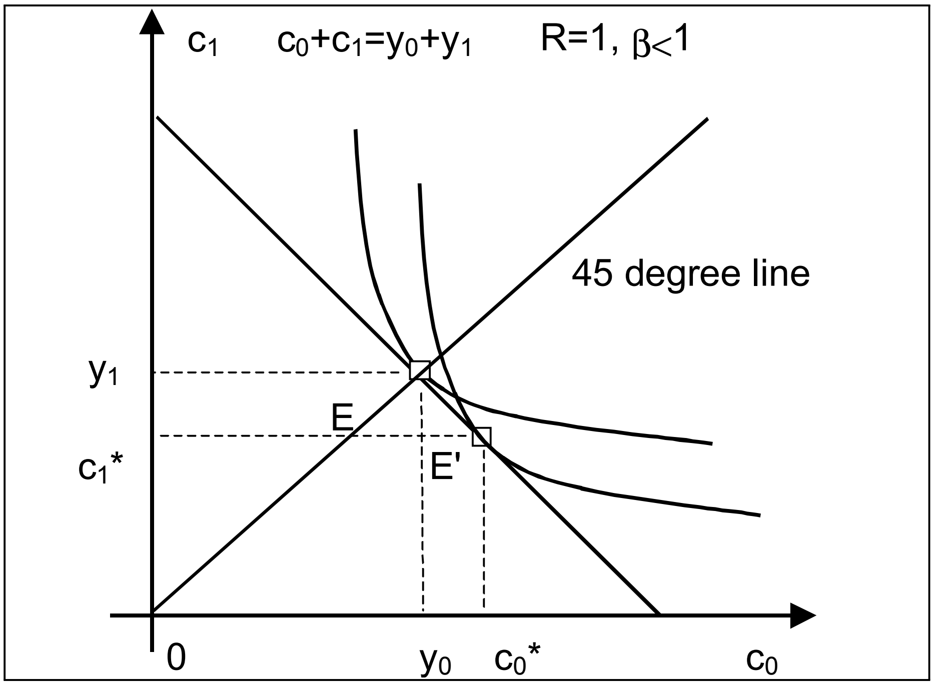

Figure 2.5. An increase in impatience toward future
(The case with $R = 1, y_0 = y_1 = y$)

To find the role of the rate of time preference for savings, assume that the household's income profile is flat and the real interest rate is 1. However, $\beta < 1$. The condition of the optimal consumption level c_0, (2.17), becomes

$$U'(c_0) = \beta U'(2y - c_0).$$

Savings are negative in this case. That is, it is optimal for the household to borrow at date 0. The effect of an increase in impatience, i.e., a decrease in β, is depicted in Figure 2.5. Point E is the optimal consumption choice when $\beta = 1$ (and $y_1 = y_0$, $R = 1$). At this point, the optimal level of savings is zero and so the optimal consumption level in each period is equal to income in that period. When the discount factor, β, falls to a level below one, the degree of impatience increases from 0 to a positive number. The household discounts future and the indifference curve rotates counter-clockwise. The new optimal consumption profile is given by point E'. Since point E' is below the 45^0 line, the household consumes more at date 0 than date 1, although the household's income profile is completely flat. Savings are negative at point E'.

We can use the same method as in the last subsection to explain why savings fall when the household becomes more impatient. Suppose that the household chooses the same consumption level in the two periods, i.e., $c_0 = c_1 = y$. Let the household reduce savings by a slightly positive amount δ and increase consumption at date 0 by the same amount. This change in the consumption profile increases utility at date 0 by $U'(y)\delta$. The income from savings decreases by δ and so the utility level at date 1 decreases by $U'(y)\delta$. Since future utility is discounted with the discount factor β, the change in the consumption profile changes intertemporal utility by $U'(y)\delta - \beta U'(y)\delta = (1-\beta)U'(y)\delta > 0$. That is, the household can increase intertemporal utility by reducing savings slightly.

Exercise 2.4.5 *Assume that the utility function is* $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$, *where* $\sigma > 0$. *With* $R = 1$, $y_1 = y_0 = y$ *and* $\beta < 1$, *show that the optimal level of savings in the two-period economy is*

$$s_0^* = -\frac{1-\beta^{1/\sigma}}{\beta^{1/\sigma}+1}y.$$

Show that, in this case, optimal savings are higher when the household is more patient.

2.4.5 General Forms of Intertemporal Utility Function

With general forms of the intertemporal utility function, the level of optimal savings is still $s_0^* = y_0 - c_0^*$ but now c_0^* is given by (2.14). Substituting $c_1 = y_1 + R(y_0 - c_0)$, (2.14) becomes

$$\frac{u_1(c_0, y_1 + R(y_0 - c_0))}{u_2(c_0, y_1 + R(y_0 - c_0))} = R. \tag{2.18}$$

Rather than trying to achieve a smooth consumption profile, the household now tries to obtain a smooth profile of marginal utility of consumption. For example, if $R = 1$, optimal savings are at such a level that makes discounted marginal utility of consumption be the same in the two periods. Starting from the optimal consumption profile, a decrease in y_1 relative to y_0 or an increase in R tends to increase savings, as in the case with an additive intertemporal utility function.

To find the effect of impatience on savings when the intertemporal utility function is general, we need to define the degree of impatience. Impatience can be best measured when the household's consumption is the same at the two dates, i.e., when $c_0 = c_1 = c$. Along this smooth consumption profile, if the household has a higher marginal utility for current consumption than for future consumption, then the household discounts future utility. That is, we can define the rate of time preference as

$$\rho \equiv \frac{u_1(c,c)}{u_2(c,c)} - 1 \tag{2.19}$$

The household discounts future along the smooth consumption profile if and only if $\rho > 0$.

The above definition makes sense. With the additive intertemporal utility function, the definition recovers the discount rate $1/\beta - 1$, as shown in the following exercise.

Exercise 2.4.6 *When the intertemporal utility function has the form (2.9), show that* $\rho = \frac{1}{\beta} - 1$.

When the intertemporal utility function is not time additive, the measure ρ can still be simple sometimes, as in the following exercise. Also, as in the time-additive case, the more patient the household is toward the future, the higher the savings.

Exercise 2.4.7 *Let the intertemporal utility function be* $u(c_0, c_1) = c_0^{\alpha} c_1^{1-\alpha}$, *where* $\alpha \in (0,1)$. *Show that the rate of time preference is* $\rho = \frac{2\alpha - 1}{1-\alpha}$. *Use this utility function to show that optimal levels of consumption and savings are as follows:*

$$c_0^* = \alpha\left(y_0 + \frac{y_1}{R}\right), \quad c_1^* = (1-\alpha)(Ry_0 + y_1), \quad s_0^* = (1-\alpha)y_0 - \frac{\alpha}{R}y_1.$$

Show that s_0^* *increases with* y_0 *for given* y_1, *increases with* R, *and decreases with* ρ.

Keywords for this chapter:

- Intertemporal trade-off, intertemporal maximization, Lagrangian method;
- Marginal utility of consumption, marginal utility of income, indifference curve; marginal rate of substitution, marginal rate of transformation;

- Optimal savings, consumption smoothing, income profile, real interest rate, rate of time preference.

Further readings:

- Dixit, A.K., *Optimization in Economic Theory*, 2nd edition, chapters 1 – 4, Oxford University Press, 1990.

- de la Fuente, A., *Mathematical Methods and Models for Economists*, chapters 7 and 8, Cambridge University Press, 2000.