The Real Business Cycle Model

Vivaldo Mendes

Department of Economics

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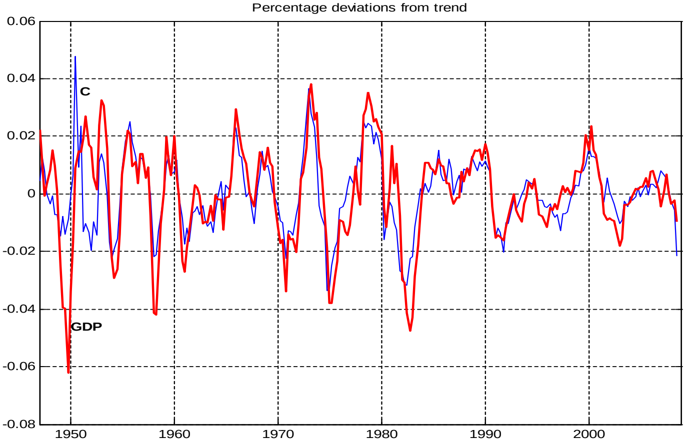
Summary

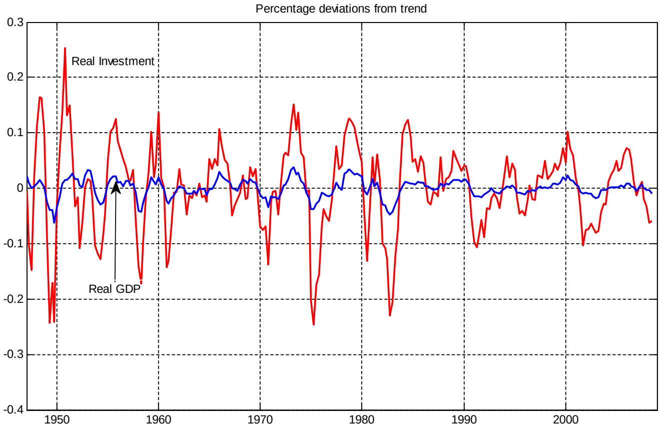
1. Major stylized facts (revisited)
2. How modern Macro explains business cycles
3. The Real Business Cycle Model: the baseline version
4. Recap: how to transform functions from levels into log differences
5. Linearizing the RBC model in the vicinity of the steady state
6. Numerical simulation of the linearized model

Major stylized facts of business cycles: a review

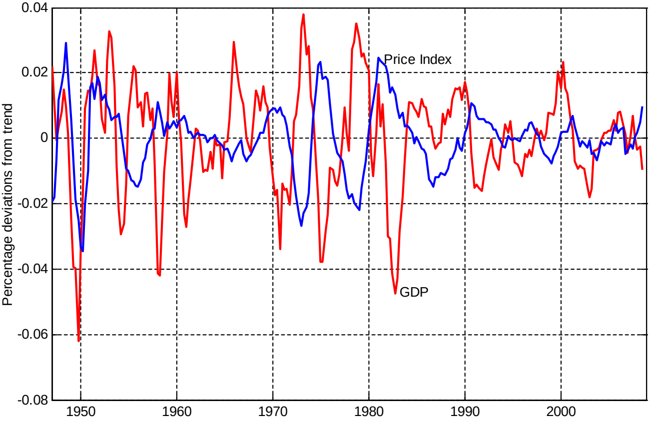
% deviations from trend: GDP vs Consumption



% deviations from trend: GDP vs Investment

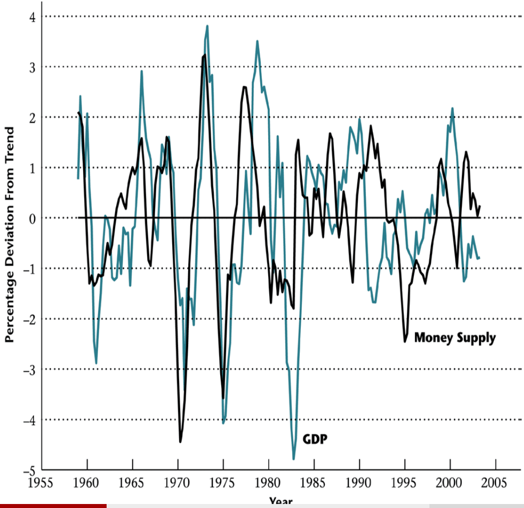


% deviations from trend: GDP vs Price Index

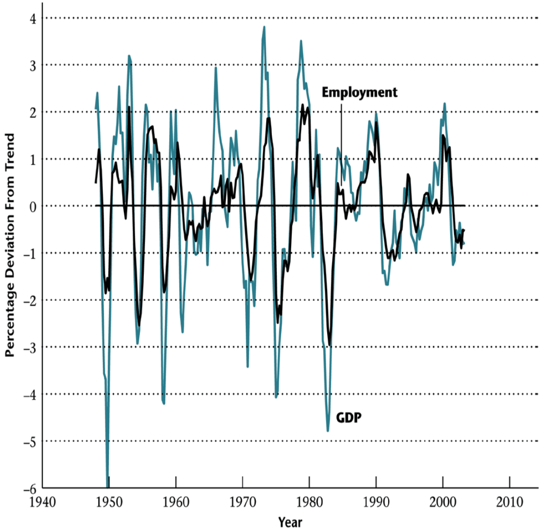


% deviations from trend: GDP vs Money Supply

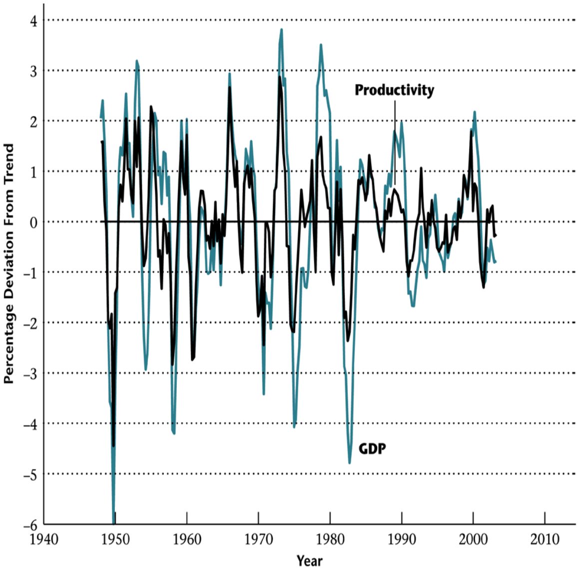
Next figures from Stephen Williamson, *Macroeconomics*, Addison-Wesley, New York, 2005.



% deviations from trend: GDP vs Employment



% deviations from trend: GDP vs Productivity



Major stylized facts of business cycles: Summary

From Stephen Williamson, *Macroeconomics*, Addison-Wesley, New York, 2005.

	Cyclicality	*Lead/Lag*	*Variability Relative to GDP*
Consumption	Procyclical	Coincident	Smaller
Investment	Procyclical	Coincident	Larger
Price Level	Countercyclical	Coincident	Smaller
Money Supply	Procyclical	Leading	Smaller
Employment	Procyclical	Lagging	Smaller
Real Wage	Procyclical	?	?
Average Labor Productivity	Procyclical	Coincident	Smaller

- **Attention: price level as countercyclical and coincident is controversial!**

How modern Macro explains business cycles

Models used to explain business cycles

1. Economies fluctuate over time
2. **Systematic facts** that need to be explained
    1. volatility
    2. Comovements
    3. Persistence (the past impacts on the present): autocorrelation
    4. How expectations affect current economic decisions
3. **Theoretical models** that have been presented to explain these facts:
    1. Market clearing models
        1. Misperception Lucas/Friedman model
        2. Coordination failures
        3. **Real business cycles**
    2. Non-Market clearing models
        1. **New Keynesian model**

Two competing models in modern macro

1. Real Business Cycles **(RBC) vs** New Keynesian Model **(NKM)**
2. **A common framework**:
    1. Dynamic General Equilibrium
    2. Stochastic shocks
    3. Quantitative (or computational): simple parables is not enough anymore
    4. Forward looking (Rational) Expectations
3. **A crucial divergence** about information and prices:
    1. complete and flexible (RBC)
    2. incomplete and sticky (NKM)
4. **Two major tools**: we need to have good knowledge of
    1. How to solve models with Rational Expectations
    2. How to optimize over time (dynamic optimization)

The RBC model: introduction

1. **The essence of the model**:
   1. Take the Solow growth model
   2. Add shocks to Total Factor Productivity (the A variable in the Solow Model)
   3. Add leisure to account for changes in hours of work
2. **A competitive equilibrium** it's about
   1. Households: preferences
   2. Firms: technology
   3. Government: policy decisions
3. **Real Factors:** preferences, technology, policy decisions are all *real factors*, that's where the name comes from (Real Business Cycles)

TFP as the fundamental mechanism

1. **The fundamental "mechanism"** of the model is shocks to Total Factor Productivity (TFP)
    1. Remember the Figure of GDP vs Productivity for the US presented above: the correlation positive and high
2. What happens if there is a **"sunny day"**(if productivity increases), or a "rainy day"?
    1. intertemporal substitution of labor and saving decisions
3. **Major result: fluctuations as an equilibrium outcome**
    1. *work harder*, when productivity is high, because wages increase as labor becomes more productive
    2. *save more*, when productivity is high, because interest rates increase as capital becomes more productive
4. Therefore, **fluctuations are not as bad as usually considered**

The Real Business Cycle Model: the baseline version

1. There are many variations on the standard RBC model[1]
2. We follow the baseline version

[1] The seminal paper is by Finn Kydland and Edward Prescott (1982), Time to Build and Aggregate Fluctuations, *Econometrica*, 50, 1345–1370).

Households: the problem

1. **Households maximize** utility over time
2. **Utility depends** on consumption (C) and hours worked (N); $u(C,N)$
3. **Intertemporal utility is discounted** by a factor β, then

$$u(\cdot) = \overbrace{\beta^0 \cdot u(C_{t+0}, N_{t+0})}^{period\ t+0} + \overbrace{\beta^1 \cdot u(C_{t+1}, N_{t+1})}^{period\ t+1} + \overbrace{\beta^2 \cdot u(C_{t+2}, N_{t+2})}^{period\ t+2} + ... \quad (1)$$

4. Notice that $\beta^0 = 1$; the discount factor is $\beta = 1/(1+r)$, where r is the subjective discount rate of future utility.

Households: with uncertainty

1. **Introducing uncertainty:** the future values of (C, N) are not known with certainty
2. **Expectations operator**: expectations operatoreq. (1) can be written at time t as

$$u(\cdot) = \underbrace{E_t}_{??} \left[u(C_t, N_t)\right] + E_t \left[\beta \cdot u(C_{t+1}, N_{t+1})\right] + ...$$

3. Notice that at t, the values of (C_t, N_t) are known: $E_t \left[u(C_t, N_t)\right] = u(C_t, N_t)$
4. The previous sum can be expressed in a more compact form

$$E_t \left[\sum_{i=0}^{\infty} \beta^i \cdot u(C_{t+i}, N_{t+i})\right]$$

Households: utility function

1. **Specific form of utility**: assume that the utility function is given by [2]

$$u(\cdot) = \frac{C^{1-\eta}}{1-\eta} - \xi N$$

   where η, ξ are parameters

2. Notice that the utility function is linear in N and nonlinear in C
3. The function that households maximize is given by

$$\max E_t \left[\sum_{i=0}^{\infty} \beta^i \cdot \left(\frac{C_{t+i}^{1-\eta}}{1-\eta} - \xi N_{t+i} \right) \right] \quad (2)$$

4. The household's behavior is caracterized: let's move to the **firm's side**

[2] **Atention**: Whelan's notation was changed (ξ instead of a), because "a" is used twice in his text (for two different meanings) and that may be confusing.

Firms: production

1. **Production**: firms produce goods and services with the following production function
$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} \tag{3}$$
Y is output, K is capital, N is labor, A is Total Factor Productivity (TFP), and α is the output/capital elasticity
2. Two relevant points:
    1. The stock of capital (K) at t **is given** by its level accumulated up to $t-1$
    2. **Constant returns to scale** with respect to the two factors that are remunerated (K, N)
3. How K, N, A are accumulated over time?

Firms: accumulation of inputs

1. **Capital**: the accumulation of K obeys

$$K_t = K_{t-1} + \underbrace{I_t - \delta K_{t-1}}_{\Delta K} = (1-\delta)K_{t-1} + I_t \tag{4}$$

   where I_t is investment and δ the depreciation rate

2. **TFP**: assume TFP does not increase over time (no trend), fluctuates around its steady state value (A^*), due to exogenous shocks (ε_t)

$$\ln A_t = (1-\rho)\ln A^* + \rho \ln A_{t-1} + \varepsilon_t \quad , \quad \rho < 1 \tag{5}$$

   1. Why logarithms $(\ln)$? To make things easier
   2. Define $a_t = \ln A_t - \ln A^*$, then (5) can be written as

$$a_t = \rho a_{t-1} + \varepsilon_t \tag{6}$$

      i.e., the log-deviation of TFP from its steady state is an AR(1) process with $\rho < 1$.

3. **Labor force:** stays constant over time.

The optimal problem for the central planner

1. There are two ways to solve for the equilibrium: a **decentralized equilibrium** and a **central planner equilibrium**
2. **A social planner** that maximizes the **objective function** subject to **a resource constraint**.
3. **The constraint** is derived from the well known national accounting identity
$$Y_t = C_t + I_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} \tag{7}$$
4. **Production** (Y_t) is affected by the level of capital (4)
$$K_t = (1-\delta)K_{t-1} + I_t$$
5. **Consolidating**: (7) and (4) can be consolidated by cancelling out I_t, we get
$$A_t K_{t-1}^{\alpha} N_t^{1-\alpha} = C_t + K_t - (1-\delta)K_{t-1}$$

The maximization of utility: the Lagrangian

1. **The Lagrangian** looks formidable but is like the "Boooo" story

$$\begin{aligned}\mathcal{L} &= E_t \left[\sum_{i=0}^{\infty} \beta^i \left\{ \left(\frac{C_{t+i}^{1-\eta}}{1-\eta} - \xi N_{t+i} \right) \right.\right. \\ &\qquad \left.\left. + \lambda_{t+i} \left(A_{t+i} K_{t+i-1}^{\alpha} N_{t+i}^{1-\alpha} + (1-\delta) K_{t+i-1} - C_{t+i} - K_{t+i} \right) \right\} \right]\end{aligned}$$

   where λ_t stands for the Lagrangian multiplier

2. **FOCs**: write the Lagrangian for two consecutive periods (as in the Solow model) and take first order conditions (FOC) with respect to C_t, K_t, N_t, λ_t

$$\partial\mathcal{L}/\partial C_t = 0, \;\; \partial\mathcal{L}/\partial K_t = 0, \;\; \partial\mathcal{L}/\partial N_t = 0, \;\; \partial\mathcal{L}/\partial \lambda_t = 0$$

3. **A small trick:** it will be useful to define the marginal value of an additional unit of capital next year (R_{t+1}) as

$$R_{t+1} \equiv \alpha \frac{Y_{t+1}}{K_t} + 1 - \delta \tag{8}$$

The Lagrangian for two consecutive periods

1. **To avoid too many symbols**: use the generic utility function $u\left(C_t, N_t\right)$ instead of $u = \frac{C_{t+i}^{1-\eta}}{1-\eta} - \xi N_{t+i}$
2. Here is the $\mathcal{L}$ function for t and $t+1$ (forget about expectations for now)

$$\begin{aligned}\mathcal{L} &= \ldots + \beta^0 \{u\left(C_t, N_t\right) + \lambda_t (A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta) K_{t-1} - C_t - K_t)\} \\ &\quad + \beta^1 \{u\left(C_{t+1}, N_{t+1}\right) + \lambda_{t+1} (A_{t+1} K_t^{\alpha} N_{t+1}^{1-\alpha} + (1-\delta) K_t - C_{t+1} -\end{aligned}$$

3. Now let's go for the two firts FOCs

$$\partial \mathcal{L} / \partial C_t = \beta^0 \left(u'_{C_t} - \lambda_t\right) = 0 \tag{9}$$

$$\partial \mathcal{L} / \partial K_t = -\beta^0 \cdot \lambda_t + \beta^1 \cdot \lambda_{t+1} \left(\alpha \cdot \underbrace{A_{t+1} K_t^{\alpha-1} N_{t+1}^{1-\alpha}}_{= Y_{t+1}/K_t} + 1 - \delta\right) = 0 \tag{10}$$

The Lagrangian for two consecutive periods

1. Here is the $\mathcal{L}$ function for t and $t+1$ again

$$\begin{aligned}\mathcal{L} &= \ldots + \beta^0 \{ u(C_t, N_t) + \lambda_t (A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta) K_{t-1} - C_t - K_t) \} \\ &\quad + \beta^1 \{ u(C_{t+1}, N_{t+1}) + \lambda_{t+1} (A_{t+1} K_t^{\alpha} N_{t+1}^{1-\alpha} + (1-\delta) K_t - C_{t+1} - \end{aligned}$$

2. Now let's go for the two last FOCs

$$\partial \mathcal{L} / \partial N_t = \beta^0 \left[u'_{N_t} + \lambda_t (1-\alpha) \underbrace{A_t K_{t-1}^{\alpha} N_t^{-\alpha}}_{= Y_t / N_t} \right] = 0 \tag{11}$$

$$\partial \mathcal{L} / \partial \lambda_t = \beta^0 \left(A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta) K_{t-1} - C_t - K_t \right) = 0 \tag{12}$$

The FOCs simplified

1. The 4 FOCs can be written as

$$
\begin{aligned}
\partial\mathcal{L}/\partial C_t &= \beta^0\left(u'_{C_t} - \lambda_t\right) = 0 \\
\partial\mathcal{L}/\partial K_t &= -\beta^0 \cdot \lambda_t + \beta^1 \cdot \lambda_{t+1}\underbrace{\left(\alpha\frac{Y_{t+1}}{K_t} + 1 - \delta\right)}_{R_{t+1}} = 0 \\
\partial\mathcal{L}/\partial N_t &= \beta^0\left[u'_{N_t} + \lambda_t(1-\alpha)\frac{Y_t}{N_t}\right] = 0 \\
\partial\mathcal{L}/\partial \lambda_t &= \beta^0\left(A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta)K_{t-1} - C_t - K_t\right) = 0
\end{aligned}
$$

2. **They can be simplified**: eliminate λ_t, λ_{t+1}. From $\partial\mathcal{L}/\partial C_t = 0$, we know that as $\beta^0 = 1$, then $u'_{C_t} - \lambda_t = 0$, that is

$$u'_{C_t} = \lambda_t \quad , \quad u'_{C_{t+1}} = \lambda_{t+1}$$

The FOCs simplified

1. Insert previous result $u^{'}_{C_t} = \lambda_t \quad , \quad u^{'}_{C_{t+1}} = \lambda_{t+1}$ into the FOC $\partial\mathcal{L}/\partial K_t$, and get the well known **Euler equation**

$$u^{'}_{C_t} = \beta(u^{'}_{C_{t+1}} R_{t+1}) \tag{13}$$

2. **Let's bring expectations back** into eq. (13)
3. The **Euler equation** with uncertainty is

$$\begin{aligned} E_t\left(u^{'}_{C_t}\right) &= E_t\left(\beta(u^{'}_{C_{t+1}} \cdot R_{t+1})\right) \\ u^{'}_{C_t} &= E_t\left(\beta(u^{'}_{C_{t+1}} \cdot R_{t+1})\right) \end{aligned}$$

4. The specific utility function can now be applied and considering that

$$u^{'}_{C_{t+i}} = \frac{\partial\left(u(\cdot,\cdot)\right)}{\partial C_{t+i}} = C_{t+i}^{-\eta}$$

5. The **Euler equation appears as**

$$\underbrace{C_t^{-\eta}} = \beta \cdot \underbrace{E_t\left(C_{t+1}^{-\eta} R_{t+1}\right)} \tag{14}$$

More on FOCs

1. Notice that from the FOCs $\partial\mathcal{L}/\partial C_t = 0$, $\partial\mathcal{L}/\partial N_t = 0$ we can get another result by cancelling out λ_t
2. Firstly,
$$\beta^t\left[u'_{N_t} - \lambda_t(1-\alpha)\frac{Y_t}{N_t}\right] = 0$$
3. As as $\beta^t \neq 0$, therefore
$$u'_{N_t} - \lambda_t(1-\alpha)\frac{Y_t}{N_t} = 0$$
4. But as $u'_{N_t} = -\xi$, and $\lambda_t = u'_{C_t}$,we get[3]
$$\frac{Y_t}{N_t} = \frac{\xi}{1-\alpha}C_t^{\eta} \tag{15}$$

[3]Note that $u'_{N_{t+i}} = \frac{\partial\left(\frac{C_{t+i}^{1-\eta}}{1-\eta} - \xi N_{t+i}\right)}{\partial N_{t+i}} = -\xi$

The maximization of utility: 4 equations x 5 variables

1. The FOCs give us **3 eq**. (8)+(14)+(15) involving **5 variables**

$$(Y_{t+i}, N_{t+i}, C_{t+i}, R_{t+i}, K_{t+i})_{i=0}^{\infty}$$

2. The system is **indeterminate**. We need two further eq. to get avoid indeterminacy
   1. the production function (eq. 3)
   2. the capital accumulation (eq. 4).
3. But these two bring another two variables into the system (A_t, I_t), which requires two further equations: (7) and (5).
4. Now the system can be solved: we have a system of **7 equations** $\times$ **7 unknowns**

$$(Y_{t+i}, N_{t+i}, C_{t+i}, R_{t+i}, K_{t+i}, A_{t+i}, I_{t+i})_{i=0}^{\infty}$$

A nonlinear model: summary

1. Our seven equations are:

$$
\begin{aligned}
R_{t+1} &\equiv \alpha\left(Y_{t+1}/K_t\right)+1-\delta && \text{(S1)}\\
C_t^{-\eta} &= \beta E_t(C_{t+1}^{-\eta}R_{t+1}) && \text{(S2)}\\
Y_t/N_t &= \left[\xi/\left(1-\alpha\right)\right]C_t^{\eta} && \text{(S3)}\\
K_t &= (1-\delta)K_{t-1}+I_t && \text{(S4)}\\
Y_t &= A_t K_{t-1}^{\alpha} N_t^{1-\alpha} && \text{(S5)}\\
C_t+I_t &= Y_t && \text{(S6)}\\
\ln A_t &= (1-\rho)\ln A^{*}+\rho\ln A_{t-1}+\varepsilon_t && \text{(S7)}
\end{aligned}
$$

2. **A nonlinear system** of stochastic difference equations (some of them are nonlinear)
3. **Solutions are extremely difficult** (if not impossible) to be obtained for these systems
4. **A trick: linearize the system** in the vicinity of the steady state. Widely used and very useful in economics

Linearization: what is it?

1. We shall **recall a number of points**:
   1. The system has 7 endogenous variables $(Y_{t+i}, N_{t+i}, C_{t+i}, R_{t+i}, K_{t+i}, A_{t+i}, I_{t+i})_{i=0}^{\infty}$
   2. In steady state, for any variable v_t, we get: $v_t = v_{t+1} = \bar{v}$
   3. The natural way to linearize an equation is to apply logs, or Δlog (first difference in logs)
   4. Remember that Δlog is approximately equal to a growth rate
2. We will apply Δlog to our system
   1. Linearization may look very complicated, but in fact it's extremely simple
3. We only need to know how to **transform the equations of the model into Δlog functions**

Linearization:
functions from levels into log differences

Transforming functions into log-differences: first case

1. **A linear function**: $Y_t = 2X_t$. Apply logs to two consecutive periods:

$$\ln Y_t = \ln 2 + \ln X_t$$
$$\ln Y_{t+1} = \ln 2 + \ln X_{t+1}$$

2. Therefore, the first difference of logs is

$$\underbrace{\ln Y_{t+1} - \ln Y_t}_{\text{growth rate: } y} = (\ln 2 + \ln X_{t+1}) - (\ln 2 + \ln X_t) = \underbrace{\ln X_{t+1} - \ln X_t}_{\text{growth rate: } x}$$

3. In this kind of function, the growth rate of Y, let's call it (y), is equal to the growth rate of X, (x)

$$y = x$$

4. Don´t forget: we use **small letters to express the growth rate of a variable**

Transforming functions into log-differences: second case

1. **A linear function of two independent variables**: $Y_t = 2X_t Z_t$.
2. Apply logs to two consecutive periods, and you will get

$$y = x + z$$

3. Prove this result yourself.

Transforming functions into log-differences: third case

1. **A power function**: $Y_t = 2X_t Z_t^{-3}$.
2. Apply logs

$$\begin{aligned} \ln Y_t &= \ln 2 + \ln X_t - 3\ln Z_t \\ \ln Y_{t+1} &= \ln 2 + \ln X_{t+1} - 3\ln Z_{t+1} \end{aligned}$$

3. Therefore, the first difference of logs is

$$\begin{aligned} \underbrace{\ln Y_{t+1} - \ln Y_t}_{\text{growth rate: } y} &= (\ln 2 + \ln X_{t+1} - 3\ln Z_{t+1}) - (\ln 2 + \ln X_t - 3\ln Z \\ &= \underbrace{\ln X_{t+1} - \ln X_t}_{\text{growth rate: } x} - 3\underbrace{(\ln Z_{t+1} - \ln Z_t)}_{\text{growth rate: } z} \end{aligned}$$

4. So this power function can be written in Δlog as

$$y = x - 3z$$

Transforming functions into log-differences: fourth case

1. The last function we need to consider is an **additive function** like

$$Y_{t+1} = X_{t+1} + Z_{t+1}$$

2. Here we can't apply logs. But there is another way
3. Firstly, multiply and divide through as follows

$$\frac{Y_{t+1}}{Y_t} Y_t = \frac{X_{t+1}}{X_t} X_t + \frac{Z_{t+1}}{Z_t} Z_t.$$

4. Now apply the following: $\frac{Y_{t+1}}{Y_t} = 1 + y$, $\frac{X_{t+1}}{X_t} = 1 + x$, $\frac{Z_{t+1}}{Z_t} = 1 + z$, and the previous eq. can be written as

$$(1 + y)\, Y_t = (1 + x)\, X_t + (1 + z)\, Z_t$$

5. Divide through by Y_t and get

$$1 + y = (1 + x)\, \frac{X_t}{Y_t} + (1 + z)\, \frac{Z_t}{Y_t}$$

Transforming functions into log-differences: fourth case (cont.)

1. Notice that the previous equation can be written as

$$1 + y = \underbrace{\left(\frac{X_t}{Y_t} + \frac{Z_t}{Y_t} \right)}_{=(X_t + Z_t)/Y_t = 1} + x\frac{X_t}{Y_t} + z\frac{Z_t}{Y_t}$$

2. Therefore, an **additive function** like $Y_{t+1} = X_{t+1} + Z_{t+1}$ can be expressed as

$$y = x\frac{X_t}{Y_t} + z\frac{Z_t}{Y_t}$$

3. Notice that if $Z = 2$, its growth rate were $z = 0$, and we would get

$$y = x\frac{X_t}{Y_t}$$

Transforming functions into log-differences: summary

1. Let's summarize our results

Variables in levels		*Variables in* Δ*logs*
$Y_t = 2X_t$	$\Leftrightarrow$	$y = x$
$Y_t = 2X_tZ_t$	$\Leftrightarrow$	$y = x + z$
$Y_t = 2X_tZ_t^{-3}$	$\Leftrightarrow$	$y = x - 3z$
$Y_{t+1} = X_{t+1} + Z_{t+1}$	$\Leftrightarrow$	$y = x\frac{X_t}{Y_t} + z\frac{Z_t}{Y_t}$
$Y_{t+1} = X_{t+1} + a$	$\Leftrightarrow$	$y = x\frac{X_t}{Y_t}$

Linearizing the RBC model in the vicinity of the steady state

Linearization

1. Transforming our system into a linear one

$$
\begin{array}{rcl}
C_t^{-\eta} = \beta E_t (C_{t+1}^{-\eta} R_{t+1}) & \Leftrightarrow & c_t = E_t c_{t+1} - \frac{1}{\eta} E_t r_{t+1} \\
Y_t / N_t = \left[\xi / \left(1 - \alpha\right) \right] C_t^{\eta} & \Leftrightarrow & n_t = y_t - \eta c_t \\
K_t = (1 - \delta) K_{t-1} + I_t & \Leftrightarrow & k_t = (1 - \delta) k_{t-1} \frac{K_{t-2}}{K_{t-1}} + i_t \frac{I_t}{K_t} \\
Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} & \Leftrightarrow & y_t = a_t + \alpha k_{t-1} + (1 - \alpha) n_t \\
C_t + I_t = Y_t & \Leftrightarrow & y_t = c_t \frac{C_t}{Y_t} + i_t \frac{I_t}{Y_t} \\
R_t \equiv \alpha \left(Y_t / K_{t-1} \right) + 1 - \delta & \Leftrightarrow & r_t = \frac{\alpha}{R_t} \frac{Y_t}{K_{t-1}} \left(y_t - k_{t-1} \right) \\
\ln A_t = (1 - \rho) \ln A^* + \rho \ln A_{t-1} + \varepsilon_t & \Leftrightarrow & a_t = \rho a_{t-1} + \varepsilon_t
\end{array}
$$

- Notice that now our system is: 7 eq. $\times$ 12 unknowns: (c, r, n, y, k, i, a) plus (K, C, Y, I, R).

Linearization: one example

1. One example. Let us solve the less simple equation of the whole set

$$R_t \equiv \alpha \left(Y_t / K_{t-1} \right) + 1 - \delta$$

2. Simplify the previous equation by assuming that

$$Z_t \equiv Y_t / K_{t-1}, \quad \text{and} \quad \phi \equiv 1 - \delta$$

3. Then we have

$$R_t \equiv \alpha Z_t + \phi$$

4. Now apply the rule discussed above and get

$$r_t = \alpha z_t \frac{Z_t}{R_t}$$

5. But as $z_t = y_t - k_{t-1}$
6. We get the final result as

$$r_t = \frac{\alpha}{R_t} \frac{Y_t}{K_{t-1}} \left(y_t - k_{t-1} \right)$$

Determining the steady state

1. In the set (K, C, Y, I, R) each variable can be determined because we are linearizing near the steady state.
2. Remember that in the vicinity of the steady state, for any v_t, we get $v_t = v_{t+1} = \bar{v}$, then $v_t / v_{t+1} = 1$.
3. Let's start with the Euler equation (eq. S2), as $C_t = C_{t+1} = \bar{C}$, then

$$\begin{aligned} C_t^{-\eta} &= \beta E_t (C_{t+1}^{-\eta} R_{t+1}) \\ 1 &= \beta E_t \left[\left(\frac{C_t}{C_{t+1}} \right)^{\eta} R_{t+1} \right] = \beta \bar{R} \\ \bar{R} &= \beta^{-1} \end{aligned}$$

4. If $\bar{R} = \beta^{-1}$, then from eq. (S1) we can obtain

$$\begin{aligned} \beta^{-1} &\equiv \alpha \left(\frac{\bar{Y}}{\bar{K}} \right) + 1 - \delta \\ \frac{\bar{Y}}{\bar{K}} &= \frac{\beta^{-1} + \delta - 1}{\alpha} \end{aligned}$$

Determining the steady state (continued)

1. As we know that $\bar{R} = \beta^{-1}$ and $\frac{\bar{Y}}{\bar{K}} = \frac{\beta^{-1}+\delta-1}{\alpha}$, then

$$\frac{\alpha}{\bar{R}}\frac{\bar{Y}}{\bar{K}} = 1 - \beta(1-\delta)$$

2. Next, from eq.(S4)

$$\begin{aligned} \bar{K} &= (1-\delta)\bar{K} + \bar{I} \\ \frac{\bar{I}}{\bar{K}} &= \delta \end{aligned}$$

3. and

$$\frac{\bar{I}}{\bar{Y}} = \frac{\frac{\bar{I}}{\bar{K}}}{\frac{\bar{Y}}{\bar{K}}} = \phi, \quad \text{for simplicity with } \phi = \frac{\alpha\beta}{\beta^{-1}+\delta-1}$$

4. and finally

$$\frac{\bar{C}}{\bar{Y}} = 1 - \frac{\bar{I}}{\bar{Y}} = 1 - \phi$$

Summary: our linearized model in the vicinity of the steady state

1. Our system of **stochastic linear difference equations** with **rational expectations** looks like: 7eq. $\times$ 7 unknowns

$$
\begin{aligned}
c_t &= E_t c_{t+1} - \frac{1}{\eta} E_t r_{t+1} \\
n_t &= y_t - \eta c_t \\
k_t &= (1-\delta) k_{t-1} + \delta i_t \\
y_t &= a_t + \alpha k_{t-1} + (1-\alpha) n_{t-1} \\
y_t &= c_t (1-\phi) + \phi i_t \\
r_t &= [1 - \beta(1-\delta)] (y_t - k_{t-1}) \\
a_t &= \rho a_{t-1} + \varepsilon_t
\end{aligned}
$$

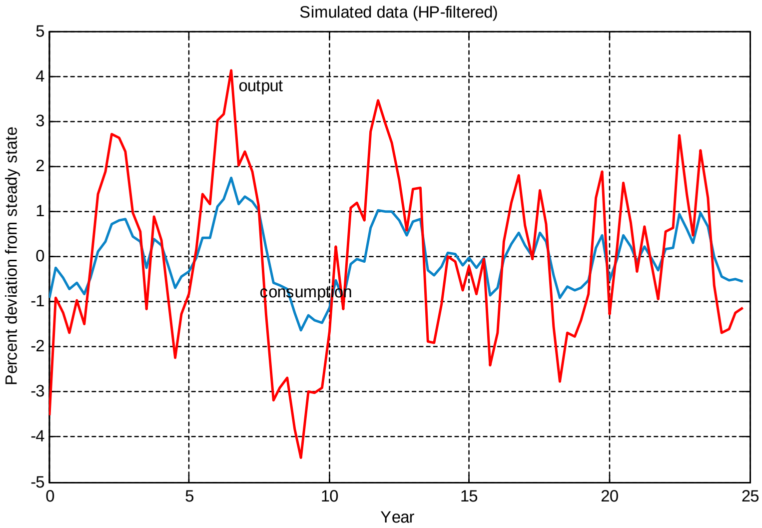
- With $\phi = \frac{\alpha\beta}{\beta^{-1}+\delta-1}$.

Numerical simulation of the linearized model

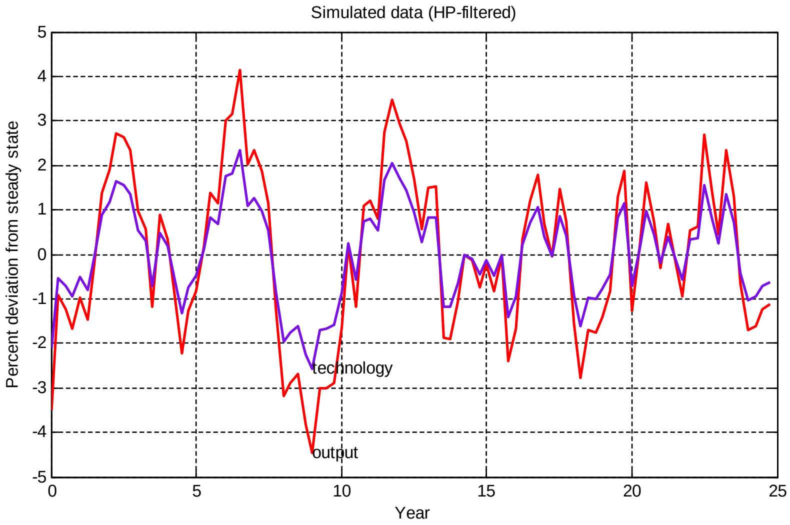
Numerical simulation

1. After a closed form solution is obtained by eliminating the expectations operators
2. If we give numbers to the parameters, we can take the model to the computer and simulate **the impact of shocks upon the endogenous variables**
3. We use a routine for Matlab developed by Harald Uhlig, now at the University of Chicago.
4. See this link to get much more variations on the RBC model taken to the computer, written by Jesus Fernandez-Villaverde (University of Pennsylvania) *www.cepremap.cnrs.fr/juillard/mambo/*
5. Calibrate the model: $\alpha = 0.4, \delta = 0.012, \rho = 0.95, \beta = 0.987, \sigma_\epsilon = 0.007$, and ξ such that $\bar{n} = 1/3$.
6. See the next figures

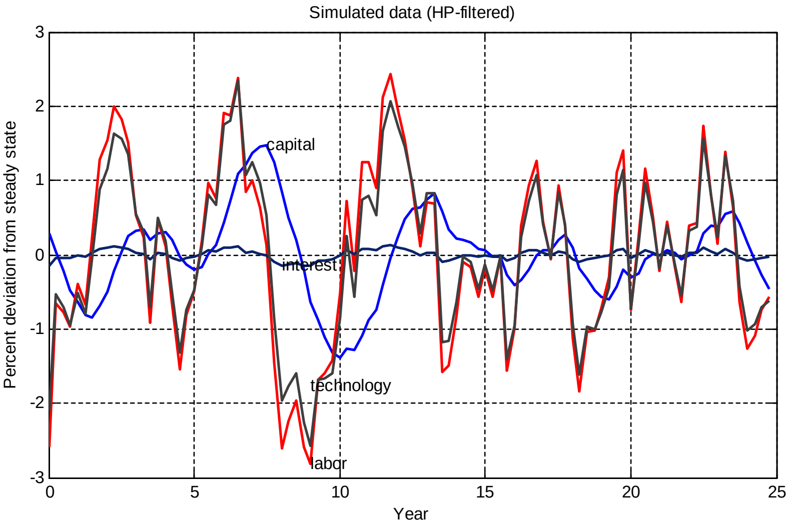
Output vs consumption



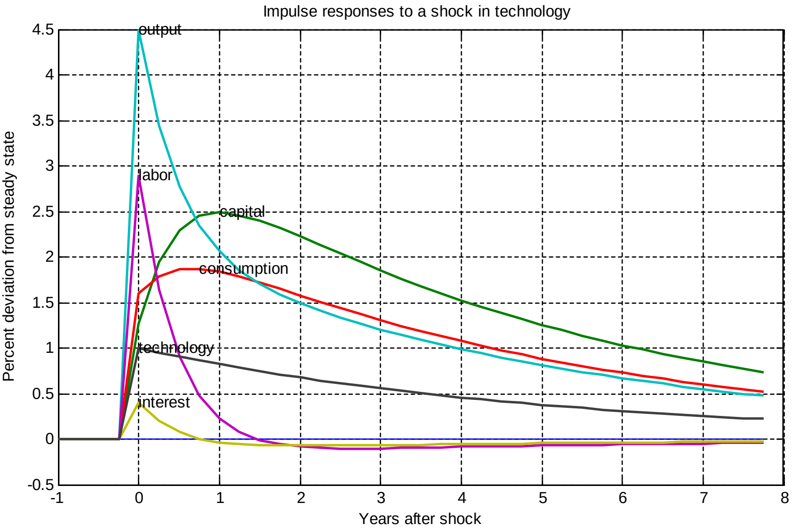
Output vs TFP



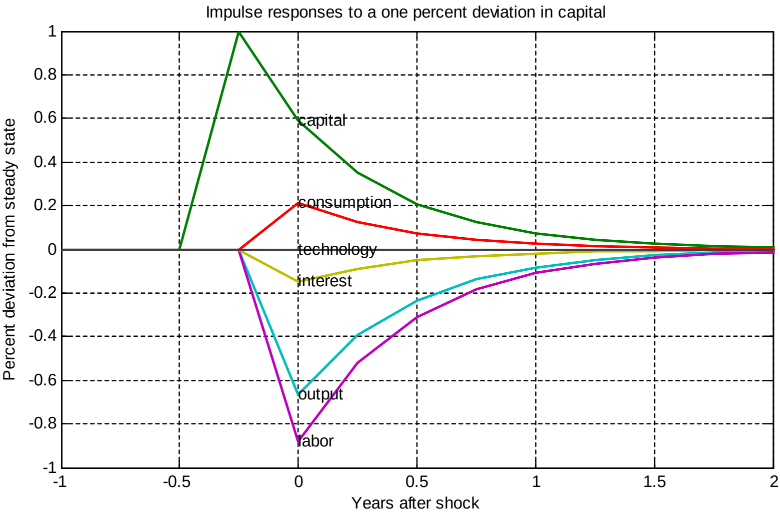
Capital, interest rate, TFP and labor



A positive technological shock



A one % increase in capital



The RBC model: shortcomings

1. **Reproduces relatively well** several stylized facts of business cycles
   1. Output is nearly as volatile as in the data (1.39% vs. 1.81%).
   2. Consumption is less volatile than output (0.44 vs. 0.74)
   3. Investment is more volatile (3 times)
   4. Persistence is high
2. It seems OK with covariances
3. **Serious problems**:
   1. Variability of hours of work is understated as well as consumption
   2. Real wages and interest rates are highly procyclical (not so in the data)
   3. Where do the negative shocks come from?
   4. No role for monetary policy
   5. Fiscal policy is of little help due to Ricardian equivalence