Linear Models with Rational Expectations

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Summary

1. From Adaptive to Rational Expectations
2. Some Evidence on Rational Expectations
3. Solving Models with Rational Expectations
4. Backward vs Forward solutions with predetermined variables
5. Backwards versus Forward solutions with forward looking variables
6. The Blanchard-Kahn decoupling method (not compulsory)

I – From Adaptive to Rational Expectations

From Adaptive to Rational Expectations

1. **Economics is different**: contrary to physics, biology, and other subjects, in economics most decisions are based on the agents's expectations
2. **Rational behavior**: humans are rational animals (they think) and so they try to optimize their well-being using all available information
3. **Two competing ways** in economics have been competing to deal with the formulation of expectations
    1. Backward-looking (or adaptive) expectations
    2. Forward looking (or rational) expectations
4. **Up to the mid 1970s**, macro theory was largely based on **adaptive expectations**
5. **From the mid 1970s onwards**, rational expectations seem to have gained predominance

Adaptive expectations

1. **Some simple heuristic rule** is used by economic agents, where the expected future value of a variable is obtained from a weighted average of its past values
    1. People collect information from the past
    2. Take into account the mistakes they made over that period
    3. Introduce this past trend into their current decisions
2. **No information about the current state of the economy**, nor of any future expected development that is likely to happen, is considered by agents
3. **Formally**, adaptive expectations were modeled as follows

$$P_t^e = P_{t-1}^e + \psi \left(P_{t-1} - P_{t-1}^e \right) \tag{1}$$

4. P_t^e is the expected value at t of the true value of a variable P_t, $\psi = 0$, myopic expectations, $0 < \psi < 1$, adaptive expectations

Adaptive expectations: solving by backward iteration

1. Equation (1) can be written to isolate P_{t-1} on the right-hand side

$$P_t^e = \psi P_{t-1} + (1-\psi)\, P_{t-1}^e \tag{2}$$

2. Now, let's get rid off P_{t-1}^e ... by **backward iteration**
3. We know that

$$P_{t-1}^e = \psi P_{t-2} + (1-\psi)\, P_{t-2}^e \tag{3}$$

4. **2nd iteration**: insert eq. (3) into eq. (2) leads to

$$\begin{aligned} P_t^e &= \psi P_{t-1} + (1-\psi)\left(\psi P_{t-2} + (1-\psi)\, P_{t-2}^e\right) \\ &= \psi P_{t-1} + \psi\,(1-\psi)\, P_{t-2} + (1-\psi)^2\, P_{t-2}^e \end{aligned}$$

Solving by repeated substitution (cont.)

1. Now let's get rid off P^e_{t-2} in the previous equation:**3rd iteration**

2. As $P^e_{t-2} = \psi P_{t-3} + (1-\psi) P^e_{t-3}$, then

$$\begin{aligned} P^e_t &= \psi P_{t-1} + \psi (1-\psi) P_{t-2} + (1-\psi)^2 \left(\psi P_{t-3} + (1-\psi) P^e_{t-3}\right) \\ P^e_t &= \psi P_{t-1} + \psi (1-\psi) P_{t-2} + \psi (1-\psi)^2 P_{t-3} + (1-\psi)^3 P^e_{t-3} \end{aligned}$$

Solving by repeated substitution (cont.)

1. At the **third iteration** backwards in time, we arrived at

$$P_t^e = \psi P_{t-1} + \psi (1-\psi) P_{t-2} + \psi (1-\psi)^2 P_{t-3} + (1-\psi)^3 P_{t-3}^e$$

2. We could go on, and on (back in time), getting rid off of P_{t-i}^e, with $i = 0, 1, 2, ...$
3. Fortunatelly, that is not necessary: we can see a pattern in our backwards solution. At the third iteration $(n = 3)$ we got

$$P_t^e = (1-\psi)^3 P_{t-3}^e + \sum_{i=0}^{3-1} \psi (1-\psi)^i P_{t-1-i}$$

4. Now, generalise to the nth iteration backwards and

The result is

$$P_t^e = (1-\psi)^n P_{t-n}^e + \sum_{i=0}^{n-1} \psi (1-\psi)^i P_{t-1-i} \tag{4}$$

Adaptive expectations: stability

1. The solution we obtained

$$P_t^e = (1-\psi)^n P_{t-n}^e + \sum_{i=0}^{n-1} \psi (1-\psi)^i P_{t-1-i}$$

2. Notice that, if $|1-\psi| > 1$, P_t^e would become explosive as time goes on. The reason is that

$$\lim_{n\to\infty} (1-\psi)^n P_{t-n}^e = \infty$$

3. The solution is stable (not explosive) if and only if

$$|1-\psi| < 1$$

4. Adaptive expectations has a stable soluition if $|1-\psi| < 1$, which is equal to

The result is

$$P_t^e = \sum_{i=0}^{n-1} \psi (1-\psi)^i P_{t-1-i} \tag{5}$$

Adaptive expectations: advantages and limitations

1. **Some advantages** have been pointed out:
    1. Expectations are in essence **subjective**, so how can we make them operational in an economic model?
    2. Thousand people ... thousand expectations
    3. It turns out that, from (5), expectations can be calculated through an objective mechanism
    4. Moreover, in a world of high uncertainty the only thing rational agents can do is to cling to some marks from the past
2. **Severe limitations**:
    1. Leads to systematic mistakes
    2. Agents are irrational: do not make use of all available information
3. **Criticisms in the 1970s** by Robert Lucas and Thomas Sargent led to a revolution in macroeconomics by promoting an alternative approach called "**rational expectations**."

Rational Expectations: Basic arguments

1. **Economic agents are fully rational**: they make use of all available information, not just past information
2. The expectations that agents have today of what may happen tomorrow have important impact on what agents do today: **forward looking expectations**
3. **Rigidities**: forward-looking dynamics naturally stem from optimizing behavior in the presence of rigidities:
    1. Had we perfect and complete information
    2. And if our hands were not tied to some constraints at any point in time,
    3. we would not need to look forward into the expected future in order to make good decisions:
4. **Natural** to use forward expectations when we want to make some decision, but unsure whether we can implement it or not

Rational Expectations: Basic arguments

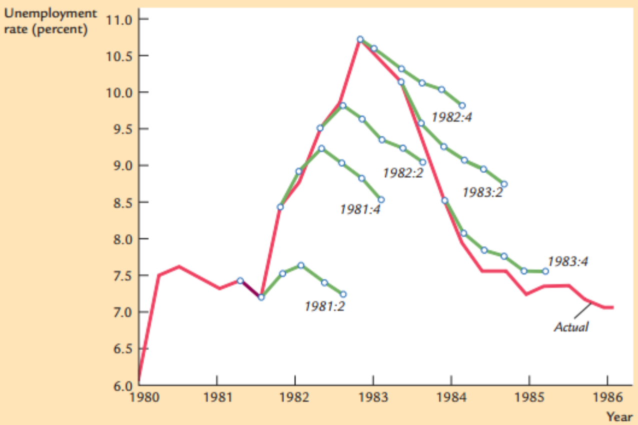
1. In modern macroeconomics, the term rational expectations means three things **(very important)**:
    1. Agents use publicly available information in an efficient manner, and therefore no systematic mistakes are produced when formulating expectations.
    2. Agents understand the structure of the model/economy and base their expectations of variables on this knowledge.
    3. Therefore, agents can forecast everything with no systematic mistakes; the only thing they cannot forecast are the **exogenous shocks** that hit the economy. These are unforcastable and unpredictable.
2. **Strong assumption**: the structure of the economy is complex and in truth nobody truly knows how everything works.
3. **How well does RE fit in reality**? Let's look at some pictures

II – Some evidence on Rational Expectations

Mistakes Forecasting the Recession of 1982

- From N. G. Mankiw (2009), *Macroeconomics*, Worth Publishers

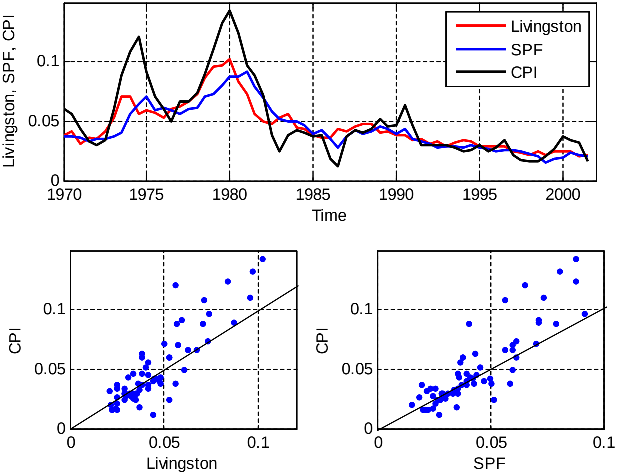
FIGURE 15-1



Rational Expectations: How do they fit the facts

1. In the following figures we present some evidence about expectations of consumer prices and interest rates
2. SPF - Survey of Professional Forecasters, Livingston Survey, and CPI (Consumer Price Index)
3. The Livinsgston Survey is the oldest continuous survey of economists' expectations; it summarizes the forecasts of economists from industry, government, banking, and academia. The Survey of Professional Forecasters is the oldest quarterly survey of macroeconomic forecasts in the United States, began in 1968 and is managed by the Federal Reserve Bank of Philadelphia since 1990.
4. Data was obtained from the Federal Reserve Bank of Philadelphia website: *http://www.philadelphiafed.org/econ/forecast/*

Survey expectations on inflation



Some points should be highlighted

1. Evidence suggests that:
    1. **For low levels of inflation**, expectations are very close to the true values of inflation
    2. **For high levels of inflation**, divergent expectations are easy to spot (systematic mistakes)
    3. Short-term expectations produce **a much smaller error** than forecasting for larger periods of time
    4. **No systematic mistakes** are visible in the short-term formulation of expectations
2. Do **Shocks** explain the rather small divergence between expectations and true values?
3. What about **heterogeneous expectations** ... (not covered here)
4. So, probably what we may have is "**Near Rational Expectations**" (George Akerlof and associates)

III – Solving Models with Rational Expectations

Models with RE: the typical problem

1. Lots of models in economics take the form

$$y_t = x_t + a \cdot E_t y_{t+1} \tag{6}$$

2. It says that y today is determined by x today and by tomorrow's expected value of y. What determines this expected value?
3. Under RE, the agents understand what's in the equation and formulate expectations in a way that is consistent with it:

$$E_t y_{t+1} = E_t x_{t+1} + a \cdot E_t \left[E_{t+1} y_{t+2} \right] \tag{7}$$

4. **We have a problem**: what is this term is the previous eq.?

$$E_t \left[E_{t+1} y_{t+2} \right]$$

5. The **Law of Iterated Expectations** gives the answer.

Models with RE: the law of iterated expectations

1. The **Law of Iterated Expectations**: today, not rational to expect to have a different expectation at $t+1$ for y_{t+2} than the one I have today.
2. Then, we get
$$E_t \left[E_{t+1} y_{t+2} \right] = E_t y_{t+2}$$
3. And eq. (7) can be written as
$$E_t y_{t+1} = E_t x_{t+1} + a \cdot E_t y_{t+2} \tag{8}$$
4. Now we can substitute eq. (8) into equation (6), and get: **2nd iteration**
$$y_t = x_t + a \cdot E_t x_{t+1} + a^2 E_t y_{t+2}$$
5. Let's iterate forward once again by substituting for $E_t y_{t+2}$: **3rd iteration**
$$y_t = x_t + a \cdot E_t x_{t+1} + a^2 E_t x_{t+2} + a^3 E_t y_{t+3}$$

Models with RE: the solution in compact form

1. At the **3rd iteration**, we obtained this result

$$y_t = x_t + a \cdot E_t x_{t+1} + a^2 E_t x_{t+2} + a^3 E_t y_{t+3}$$

2. This can be written in more useful (compact) form as

$$y_t = \sum_{i=0}^{2} a^i E_t x_{t+i} + a^3 E_t y_{t+3}$$

Notice that when $i = 0$ we get $\underbrace{a^i}_{=1} E_t x_{t+i} = E_t x_t = x_t$

3. Generalizing for the nth iteration

$$y_t = \sum_{i=0}^{n-1} a^i E_t x_{t+i} + a^n E_t y_{t+n}$$

Models with RE: a crucial assumption

1. We have just got this result

$$y_t = \sum_{i=0}^{n-1} a^i E_t x_{t+i} + a^n E_t y_{t+n} \tag{9}$$

2. Usually, it is assumed that $\mid a \mid < 1$.**This assumption makes sense** as we will see later
3. This assumpion implies

$$\lim_{n \to \infty} a^n E_t y_{t+n} = 0$$

4. So the solution to equation (19) is

$$y_t = \sum_{i=0}^{n-1} a^i E_t x_{t+i}$$

5. This solution underlies the logic of most modern macroeconomic models.

An example of RE: Asset Pricing

1. **Consider a financial asset**:
   1. Bought today at price P_t, and pays a dividend of D_t per period.
   2. Assume a close substitute asset (e.g., a deposit with interest) that yields a safe rate of return given by r.
2. **A risk neutral investor** holds the asset if both assets get the same expected rate of return

$$\frac{D_t + E_t P_{t+1}}{P_t} = 1 + r$$

3. Solve for P_t, in order to simplify define $\phi = (1 + r)$, and get

$$P_t = (1/\phi) D_t + (1/\phi) E_t P_{t+1}$$

An example of RE: Asset Pricing (cont.)

1. By the usual method of repeated substitution, iterate forward
2. Take into account that, as $\phi = (1 + r) > 1$, then

$$|1/\phi| < 1$$

3. And the final solution, at the nth iteration, is given by

$$P_t = \sum_{i=0}^{n-1} (1/\phi)^{i+1} E_t D_{t+i}$$

4. **This is a very important result in finance**: asset prices should be equal to the discounted present-value sum of expected future dividends.

IV – Backward vs Forward solutions with predetermined variables

Forward iteration

1. Let's start with a model with **no forward looking expectations**
2. Assume that y_t is a state variable (or predetermined) and x_t is an exogenous variable

$$y_t = ay_{t-1} + x_t$$

3. Let's find a solution by **forward iteration**

$$\begin{aligned}
y_1 &= a \cdot y_0 + x_1 \\
y_2 &= a(ay_0 + x_1) + x_2 \\
y_3 &= a[a(ay_0 + x_1) + x_2] + x_3 \\
&= a^3 y_0 + a^2 x_1 + ax_2 + x_3 \\
&\quad \ldots
\end{aligned}$$

Forward iteration: the stability problem

1. At the n iteration, we get

$$y_t = a^n y_0 + \sum_{i=0}^{n-1} a^i x_{t-i} \tag{10}$$

2. y_t is a function of the past realizations of x and the initial condition y_0

The stability problem:

1. If $\mid a \mid < 1$: we get a stable solution because $a^n y_0 \longrightarrow 0$, when $n \longrightarrow \infty$
   1. Acceptable solution, because y_t does not explode in finite time
2. If $\mid a \mid > 1$: there is no stable solution because $a^n y_0 \longrightarrow \infty$, when $n \longrightarrow \infty$
   1. Bad solution because y_t explodes in finite time
   2. Counterintuitive: the weights on past values of x_t in the forward solution will explode

Summary

The result : predetermined variables + iterating forward

$$y_t = a^n y_0 + \sum_{i=0}^{n-1} \left(a^i\right) x_{t-i} \tag{11}$$

The stability problem

1. If $\mid a \mid < 1$: stable solution
2. If $\mid a \mid > 1$: no stable solution
3. If $\mid a \mid < 1$ and x_t is not explosive, then y_t is also not explosive
    1. The value of y_t changes when x_t changes (more on this later)

Backward iteration

1. **Let's now** try the **backward solution**
2. Rewrite the equation $y_t = ay_{t-1} + x_t$ such that

$$y_{t-1} = \frac{1}{a}y_t - \frac{1}{a}x_t$$

3. Starting from some T in the future, and going backwards in time, we get

$$\begin{aligned} y_{T-1} &= (1/a) \cdot y_T - (1/a) \cdot x_T \\ y_{T-2} &= (1/a)^2 \cdot y_T - (1/a)^2 \cdot x_T - (1/a) \cdot x_{T-1} \\ y_{T-3} &= (1/a)^3 \cdot y_T - (1/a)^3 \cdot x_T - (1/a)^2 \cdot x_{T-1} - (1/a) \cdot x_{T-2} \\ & \ldots \end{aligned}$$

4. And at the nth iteration backwards, we end up with the result

$$y_t = (1/a)^n \cdot y_T - \sum_{i=0}^{n-1} (1/a)^{1+i} \cdot x_{t+1+i} \tag{12}$$

Backward iteration: the stability problem

1. We obtained the result

$$y_t = (1/a)^n \cdot y_T - \sum_{i=0}^{n-1} (1/a)^{1+i} \cdot x_{t+1+i} \tag{13}$$

1. If $\mid a \mid > 1$: we get a stable solution because $(1/a)^n y_T \longrightarrow 0$, when $n \longrightarrow \infty$
   1. Acceptable solution, because y_t does not explode in finite time
2. If $\mid a \mid < 1$: there is **no** stable solution because $(1/a)^n y_T \longrightarrow \infty$, when $n \longrightarrow \infty$
   1. Bad solution because y_t explodes in finite time
   2. Counterintuitive: the weights on future values of x_t in the backward solution will explode

Summary

The result: predetermined variables + iterating backwards

$$y_t = (1/a)^n \cdot y_T - \sum_{i=0}^{n-1} (1/a)^{1+i} \cdot x_{t+1+i} \tag{14}$$

1. If $\mid a \mid > 1$: stable solution
2. If $\mid a \mid < 1$: no stable solution
3. If $\mid a \mid > 1$ and x_t is not explosive, then y_t is also not explosive
   1. The value of y_t changes when x_t changes (more on this later)

Summary: backward vs forward iteration with predetermined variables

1. **Both are correct solutions** to the first-order difference equation
2. **Except that:**
    1. Forward solution: we need to know the value of y_0
    2. Backward solution, we need to know y_T .
3. Dependence on the **value of the parameter** a.
    1. $\mid a \mid < 1$: choose **forward solution**
    2. $\mid a \mid > 1$: choose **backward solution**

V – Backwards versus Forward solutions with forward looking variables

Models with forward looking variables

1. Be very careful with the value of a, when iterating forward or backwards in this case as well.
2. Assume that y_t is a **control variable** (like an interest rate chosen by the Central Bank, or public expenditure decided by the Government),
3. And x_t is our exogenous variable
4. **Introduce forward looking expectations**, and for the sake of simplicity assume that *the expectations term is on the left hand side*

$$E_t y_{t+1} = a y_t + x_t$$

5. We may have two possible cases

$$\mid a \mid > 1 \quad ; \quad \mid a \mid < 1$$

6. Let's see what happens with forward and backward iteration

Forward iteration

1. Consider our forward looking expectations equation

$$E_t y_{t+1} = a y_t + x_t$$

2. Rewrite and **iterate forward**

$$y_t = (1/a)\, E_t y_{t+1} - (1/a)\, x_t$$

3. Get rid of the term $E_t y_{t+i}$ by repeated substitution

$$\begin{aligned} E_t y_{t+1} &= (1/a)\, E_t y_{t+2} - (1/a)\, E_t x_{t+1} \\ E_t y_{t+2} &= (1/a)\, E_t y_{t+3} - (1/a)\, E_t x_{t+2} \\ &\ldots \end{aligned}$$

4. And the result comes (determined at the *nth* iteration)

$$y_t = (1/a)^n E_t y_{t+n} - \sum_{i=0}^{n-1} (1/a)^{i+1} E_t x_{t+i}$$

Forward iteration: the stability problem

The result: forward looking variables + iterating forward

$$y_t = (1/a)^n E_t y_{t+n} - \sum_{i=0}^{n-1} (1/a)^{i+1} E_t x_{t+i}$$

The stability problem

1. If $| a | > 1$: we get a stable solution because $(1/a)^n \to 0$, when $n \to \infty$
   1. Acceptable solution, because y_t does not explode in finite time
2. If $| a | < 1$: there is **no** stable solution because $(1/a)^n \to \infty$, when $n \to \infty$
   1. Bad solution because y_t explodes in finite time
   2. Counterintuitive: the weights on the expected future values of x_t in the forward solution will explode

Summary

The result: forward looking variables + iterating forward

$$y_t = (1/a)^n E_t y_{t+n} - \sum_{i=0}^{n-1} (1/a)^{i+1} E_t x_{t+i} \tag{15}$$

1. If $\mid a \mid > 1$: stable solution
2. If $\mid a \mid < 1$: **no** stable solution
3. If $\mid a \mid > 1$ and x_t is not explosive, then y_t is also not explosive

$$y_t = - \sum_{i=0}^{n-1} (1/a)^{i+1} E_t x_{t+i}$$

1. The value of y_t changes when $E_t x_{t+i}$ changes
2. Later, we will see what happens if x_t is a stochastic process

Backward iteration

1. We know that if $|a| < 1$, the forward-iteration does not secure a stable solution.
2. Now lets try a **backward solution** to

$$E_t y_{t+1} = a y_t + x_t \tag{16}$$

3. Define a new variable giving the value of the **forecasting error**

$$z_{t+1} = y_{t+1} - E_t y_{t+1}$$

4. Rewrite eq. (16) such that the process comes as

$$a y_t = \underbrace{y_{t+1} - z_{t+1}}_{E_t y_{t+1}} - x_t$$

5. **Moving time backwards one period** in the previous equation and solving for y_t, we get

$$y_t = a y_{t-1} + z_t + x_{t-1}$$

Backward iteration (continued)

1. The equation that we have to solve is

$$y_t = ay_{t-1} + z_t + x_{t-1}$$

2. Solving backwards in time, such that

$$\begin{aligned} y_{t-1} &= ay_{t-2} + z_{t-1} + x_{t-2} \\ y_{t-2} &= ay_{t-3} + z_{t-2} + x_{t-3} \\ y_{t-3} &= .... \\ ... &= ... \end{aligned}$$

3. From the repeated substitution process we get

$$y_t = a^n y_0 + \sum_{i=1}^{n-1} \left(a^i\right) z_{t-i} + \sum_{i=0}^{n-1} \left(a^i\right) x_{t-i-1}$$

4. This process is stable iff $a^n y_0$ does not explode when $n \rightarrow \infty$, that is iff $\mid a \mid < 1$.

Summary: backward vs Forward iteration with forward looking variables

1. **Both are correct solutions** to the first-order difference equation with rational expectations
2. **Except that:**
   1. Forward solution: we need to know the value of y_0
   2. Backward solution, we need to know y_T .
3. Dependence on the **value of the parameter** a.
   1. $\mid a \mid > 1$: choose **forward solution**
   2. $\mid a \mid < 1$: choose **backward solution**

Indeterminacy

1. If $| a |< 1$, there is another point that is extremely important: **INDETERMINACY!**
2. Imagine that the "fundamentals" of the economy x_t are given. We can have as many solutions to y_t as the different number of shocks that may arise, because of the expectation term $\sum_{i=1}^{n-1} \left(a^i\right) z_{t-i}$ in the solution.
3. In this case we say that we have a continuum of solutions for any given future path for x_t. That's indeterminacy, sometimes also called **SUNSPOTS** or **ANIMAL SPIRITS.**
4. The **fundamentals** of the economy (x_t) are not the only factors that drive the final outcome: so, subjective beliefs about possible shocks may move the economy out of the path that is given by the fundamentals.
5. If $| a |> 1$, we should adopt a forward iterating solution, and we get a stable value for y_t. **NO INDETERMINACY!** The value of y_t will depend only on the fundamentals: the future path of x_t.

IV – The Blanchard-Kahn decoupling method

Methods to solve RE Models

1. There are a large number of techniques presented over the past to solve this problem of a linear dynamic economic with forward looking expectations
2. The most used techniques are
    1. Repeated substitution
    2. Method of undetermined coefficients
    3. The Blanchard-Kahn decoupling method
    4. The linear projection on observables
3. We will make use of the Blanchard-Kahn method throughout the remaining part of this course
4. As most of the models in economics are multivariate, in order to save on time, we revert now to matrices to deal with the problems of stability and indeterminacy in RE models.

Stability in Matrix form

1. **The main message** from our previous exercise is:
    1. Avoid explosive solutions,
    2. Even if you have a stable solution, be worried about multiple stable solutions (indeterminacy).
2. **Iterating forward**: to obtain a unique and stable solution we need
    1. $\mid a \mid > 1$ when dealing with a **forward looking** variable
    2. $\mid a \mid < 1$ when dealing with a **predetermined** variable
3. In the multivariate case, similar conditions apply to the eigenvalues of the following system

$$\begin{aligned} \mathbf{A} \cdot \mathbf{E}_t \mathbf{z}_{t+1} &= \mathbf{B} \cdot \mathbf{z}_t \\ \mathbf{E}_t \mathbf{z}_{t+1} &= \underbrace{\mathbf{A}^{-1} \cdot \mathbf{B}}_{=W} \cdot \mathbf{z}_t \end{aligned}$$

Stability in Matrix form (continued)

1. In matrix form

$$
\begin{aligned}
\mathbf{A} \cdot \mathbf{E}_t \mathbf{z}_{t+1} &= \mathbf{B} \cdot \mathbf{z}_t \\
\mathbf{E}_t \mathbf{z}_{t+1} &= \underbrace{\mathbf{A}^{-1} \cdot \mathbf{B}}_{=\mathbf{W}} \cdot \mathbf{z}_t
\end{aligned}
$$

2. If the number of **eigenvalues** (λ_i) of $\mathbf{W}$ that lie outside the unit circle $(\mid \lambda_i \mid > 1)$
   1. is equal to the number of forward-looking variables, there **exists a unique and stable solution**
   2. is larger than the number of forward-looking variables there is **no stable solution**
   3. is lower than the number of forward-looking variables there is **an infinity of solutions**

Some notes on Matrices and Eigenvalues

1. Assume we have a 2-dimensional system as follows

$$\begin{aligned} x_{t+1} &= ax_t + by_t \\ y_{t+1} &= cx_t + dy_t \end{aligned}$$

   where a, b, c, d are parameters

2. This system can be written in matricial form as follows

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

3. In matricial form we may write

$$\mathbf{x}_{t+1} = \mathbf{A} \cdot \mathbf{x}_t$$

4. The Jacobian of matrix $\mathbf{A}$ is given by $(J = \mathbf{A})$, because our system is linear

$$J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

How to obtain the eigenvalues

1. Construct the characteristic matrix and make it equal to the characteristic polynomial

$$[J - \lambda I] = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

2. It is easy to obtain the following result

$$\det(J - \lambda I) = (a-\lambda)(d-\lambda) - bc$$

3. We get a quadratic expression with λ as the argument, and the eigenvalues can be obtained by setting this expression equal to zero
4. As matrix A is of dimension 2, we get two possible values for lambda, (λ_1, λ_2). These are the eigenvalues for this system.
5. If $\mid \lambda_1, \lambda_2 \mid < 1$, the dynamics of our system is stable
6. If $\mid \lambda_1 \mid < 1$ and $\mid \lambda_2 \mid > 1$, we have saddle path stability
7. If $\mid \lambda_1, \lambda_2 \mid > 1$, we get explosive behavior

Back to the stability problem of RE

1. We wrote our RE system as

$$\mathbf{E}_t \mathbf{z}_{t+1} = \mathbf{A}^{-1} \cdot \mathbf{B} \cdot \mathbf{z}_t = \mathbf{W} \cdot \mathbf{z}_t \tag{17}$$

2. **Example**: take a system of dimension 3, for example, $\mathbf{z}_t$ contains:
   1. 2 predetermined (or state) variables
   2. 1 control (or forward looking) variable
3. **Iterate the system forwards and avoid explosive behavior:** in order to obtain stability we need to remember that
4. **From the univariate case** stability implies:
   1. $(| a |> 1)$ when dealing with a **forward looking** variable
   2. $(| a |< 1)$ when dealing with a **predetermined** variable
5. **By analogy: in this multivariate example**, a unique and stable solution requires:
   1. $| \lambda_1 |> 1,$ for the forward looking variable
   2. $| \lambda_2, \lambda_3 |< 1,$ for the 2 predetermined variables
6. This is what we mean when we want a stable and unique solution to RE models.

The basic idea behind the B-K method

1. Apply the Jordan decomposition to transform our complicated models into two distinct blocks:
   1. The block including only predetermined variables (x_t)
   2. The other block including only forward looking variables $(E_t y_{t+1})$
2. Then we can apply the following strategy:
   1. Iterate forward
   2. Firstly, the predetermined block
   3. Then, the forward looking block.
3. And we get this type of results

$$\begin{tikzcd} Shocks: v(t) \arrow[dr] & & \\ & x(t) \arrow[r] & y(t) \\ x(t-1) \arrow[ur] & & \end{tikzcd}$$

Blanchard, O., and C.M. Kahn. (1980). The solution of linear difference models under rational expectations. *Econometrica* 48(5), 1305–1311.

The Jordan decomposition

1. Compute the Jordan canonical form (also called Jordan normal form) of a symbolic or numeric matrix A
2. Our model comprises: a set of predetermined variables (x_t), a set of forward looking variables (y_t), and a set of exogenous shocks (v_t)
3. Write the model in state space form

$$\left[\begin{array}{c} w_{t+1} \\ E_t y_{t+1} \end{array}\right] = A \left[\begin{array}{c} w_t \\ y_t \end{array}\right] + Bv_{t+1} \tag{18}$$

4. The Jordan decomposition of (A)

$$A = P\Lambda P^{-1}$$

5. Λ is a diagonal matrix with the **eigenvalues** of A along its leading diagonal and zeros in the remaining entries.
6. P contains the inverse matrix of the generalized **eigenvectors** of A as columns

The model with the Jordan decomposition

1. Apply the decomposition

$$\begin{bmatrix} w_{t+1} \\ E_t y_{t+1} \end{bmatrix} = P\Lambda P^{-1} \begin{bmatrix} w_t \\ y_t \end{bmatrix} + Bv_{t+1} \tag{19}$$

2. Multiply both sides by P^{-1}

$$P^{-1} \begin{bmatrix} w_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \Lambda P^{-1} \begin{bmatrix} w_t \\ y_t \end{bmatrix} + \underbrace{P^{-1}B}_{=R} \cdot v_{t+1} \tag{20}$$

The model with the Jordan decomposition

1. Partition P^{-1} and Λ to get

$$\underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w_{t+1} \\ E_t y_{t+1} \end{bmatrix}}_{Et \begin{bmatrix} \tilde{w}_{t+1} \\ \tilde{y}_{t+1} \end{bmatrix}} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w_t \\ y_t \end{bmatrix}}_{\begin{bmatrix} \tilde{w}_t \\ \tilde{y}_t \end{bmatrix}} + \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} v_{t+1} \tag{21}$$

2. So our transformed model looks much easier now

$$\begin{bmatrix} \tilde{w}_{t+1} \\ E_t \tilde{y}_{t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \tilde{w}_t \\ \tilde{y}_t \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} v_{t+1}$$

The two decoupled blocks

Transformed model written down as a set of decoupled equations

$$\tilde{w}_{t+1} = \Lambda_1 \tilde{w}_t + R_1 v_{t+1} \qquad \text{(Stable block)}$$

$$E_t \tilde{y}_{t+1} = \Lambda_2 \tilde{y}_t + R_2 v_{t+1} \qquad \text{(Unstable block)}$$

1. We can now apply our old strategy
    1. Solve the unstable transformed block forward and get: $\tilde{y}_t^*$
    2. Solve the stable transformed block backwards and get: $\tilde{w}_t^*$
2. Insert the results back into the original problem

Solving the unstable block

1. Iterating forward this block, we get
$$E_t \tilde{y}_{t+n} = (\Lambda_2)^n \tilde{y}_t$$
2. If we have
$$|\Lambda_2| > 1$$
3. Then, the only stable solution will be
$$\tilde{y}_t^* = 0, \ \forall t$$
4. Now from our definition in eq. (21), we know that
$$\tilde{y}_t^* = P_{21} \cdot w_t^* + P_{22} \cdot y_t^* = 0$$
5. From which
$$y_t^* = \left[-P_{22}^{-1} P_{21} \right] \cdot w_t^* \qquad (22)$$
6. **Notice that this our old result:** forward looking variables depending upon predetermined ones.

Solving the stable block

1. Iterating forward this block, we get

$$\tilde{w}_{t+n} = (\Lambda_1)^n \tilde{w}_t \quad , \qquad |\Lambda_1| < 1$$

2. If we assume that

$$|\Lambda_1| < 1$$

3. The process is stable, and from eq. (21), we get

$$\tilde{w}_t^* = P_{11} \cdot w_t^* + P_{12} \cdot y_t^* \tag{23}$$

4. Now insert eq. (22) into (23), and get

$$\tilde{w}_t^* = \underbrace{\left[P_{11} - P_{12} P_{22}^{-1} P_{21} \right]}_{D} \cdot w_t^* \tag{24}$$

Solving the stable block (cont.)

1. But as from eq. (Stable block), we have

$$\tilde{w}_{t+1} = \Lambda_1 \tilde{w}_t + R_1 v_{t+1}$$

2. And as from eq.(24) we have

$$\tilde{w}_t^* = D \cdot w_t^*$$

3. Then

$$D \cdot w_{t+1}^* = \tilde{w}_{t+1}^* \qquad \tilde{w}_t^* = D \cdot w_t^*$$

$$\tilde{w}_{t+1} = \Lambda_1 \tilde{w}_t + R_1 v_{t+1}$$

4. From which we finally get

$$w_{t+1}^* = \left[D^{-1} \Lambda_1 D \right] w_t^* + \left[D^{-1} R_1 \right] v_{t+1} \tag{25}$$

Summarizing

1. Write down your model in state space form
2. Apply the Jordan decomposition
3. Decouple the system into two blocks
4. Make sure one eigenvalue is larger than 1 in modulus, the other lower than 1 in modulus.
5. End up with the two fundamental results

$$y_t^* = \left[-P_{22}^{-1}P_{21}\right] \cdot w_t^*$$

$$w_{t+1}^* = \left[D^{-1}\Lambda_1 D\right] w_t^* + \left[D^{-1}R_1\right] v_{t+1}$$

with $D = P_{11} - P_{12}\left(P_{22}\right)^{-1} P_{21}$

V – Bibliography

Bibliography

Karl E. Whelan (2007). "Solution Methods for Rational Expectations Models", Lecture Notes, University College Dublin .

A small text very good for the study of this point. Notice that some parts of this text are not of compulsory reading.

David Peel (2005). "Advanced Macroeconomics" (Chapter 2), unpublished manuscript, Lancaster University, UK .

It's a long text that can be used as complementary studying material, but never as compulsory reading.