Linear Models with Rational Expectations

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Summary

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I – From Adaptive to Rational Expectations
Economics is different: contrary to physics, biology, and other subjects, in economics most decisions are based on the agents’s expectations.

Two competing ways in economics have been competing to deal with the formulation of expectations:

1. Backward-looking (or adaptive) expectations
2. Forward looking (or rational) expectations

Up to the mid 1970s, macro theory was largely based on adaptive expectations.

From the mid 1970s onwards, rational expectations seem to have gained predominance.
Adaptive expectations

1. **Some simple heuristic rule** is used by economic agents, where the expected future value of a variable is obtained from a weighted average of its past values:
   - People collect information from the past
   - Take into account the mistakes they made over that period
   - Introduce this past trend into their current decisions

2. **No information about the current state of the economy**, nor of any future expected development that is likely to happen, is considered by agents.

3. **Formally**, adaptive expectations were modeled as follows:

   \[ P^e_t = P^e_{t-1} + \alpha (P_{t-1} - P^e_{t-1}) \]  

   (1)

4. \( P^e_t \) is the expected value at \( t \) of the true value of a variable \( P_t \), \( \alpha = 0 \), myopic expectations, \( 0 < \alpha < 1 \), adaptive expectations.
Adaptive expectations: solving by backward iteration

1. Equation (1) can be written to isolate $P_{t-1}$ on the right-hand side

   $$P_t^e = \alpha P_{t-1} + (1 - \alpha) P_{t-1}^e$$  \hspace{1cm} (2)

2. Now, let’s get rid of $P_{t-1}^e$ ... by backward iteration

3. We know that

   $$P_{t-1}^e = \alpha P_{t-2} + (1 - \alpha) P_{t-2}^e$$  \hspace{1cm} (3)

4. 2nd iteration: insert eq. (3) into eq. (2) leads to

   $$P_t^e = \alpha P_{t-1} + (1 - \alpha) (\alpha P_{t-2} + (1 - \alpha) P_{t-2}^e)$$
   $$= \alpha P_{t-1} + \alpha (1 - \alpha) P_{t-2} + (1 - \alpha)^2 P_{t-2}^e$$
1. Now let's get rid off $P^e_{t-2}$ in the previous equation: 3rd iteration

2. As $P^e_{t-2} = \alpha P^e_{t-3} + (1 - \alpha) P^e_{t-3}$, then

$$P^e_t = \alpha P^e_{t-1} + \alpha (1 - \alpha) P^e_{t-2} + (1 - \alpha)^2 (\alpha P^e_{t-3} + (1 - \alpha) P^e_{t-3})$$

3. At the third iteration backwards in time, we get

$$P^e_t = \alpha P^e_{t-1} + \alpha (1 - \alpha) P^e_{t-2} + \alpha (1 - \alpha)^2 P^e_{t-3} + (1 - \alpha)^3 P^e_{t-3}$$
Solving by repeated substitution (cont.)

1. At the **third iteration** backwards in time, we got

   \[ P_t^e = \alpha P_{t-1} + \alpha (1 - \alpha) P_{t-2} + \alpha (1 - \alpha)^2 P_{t-3} + (1 - \alpha)^3 P_{t-3}^e \]

2. We could go on, and on backwards in time ...

3. Fortunately, that is not necessary. We can see a pattern in our backwards solution at the third iteration \((n = 3)\)

   \[ P_t^e = (1 - \alpha)^3 P_{t-3}^e + \sum_{i=0}^{3-1} \alpha (1 - \alpha)^i P_{t-1-i} \]

4. Now, generalize to the \(n\)th iteration backwards and

   \[ P_t^e = (1 - \alpha)^n P_{t-n}^e + \sum_{i=0}^{n-1} \alpha (1 - \alpha)^i P_{t-1-i} \]

   \((4)\)

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Adaptive expectations: stability

1. The solution we obtained

\[ P_t^e = (1 - \alpha)^n P_{t-n}^e + \sum_{i=0}^{n-1} \alpha (1 - \alpha)^i P_{t-1-i} \]

2. Notice that, if \(|1 - \alpha| > 1\), \(P_t^e\) would become explosive as time goes on. The reason is that

\[ \lim_{n \to \infty} (1 - \alpha)^n P_{t-n}^e = \infty \]

3. The solution is stable (not explosive) if and only if

\[ |1 - \alpha| < 1 \]

4. Adaptive expectations has a stable solution if \(|1 - \alpha| < 1\), which is

\[ P_t^e = \sum_{i=0}^{n-1} \alpha (1 - \alpha)^i P_{t-1-i} \quad (5) \]
Adaptive expectations: advantages and limitations

① **Some advantages** have been pointed out:
   ① It turns out that, from (5), expectations can be calculated through an objective mechanism
   ② Moreover, in a world of high uncertainty the only thing rational agents can do is to cling to some marks from the past

② **Severe limitations**:
   ① Leads to systematic mistakes
   ② Agents are irrational: do not make use of all available information

③ **Criticisms in the 1970s** by Robert Lucas and Thomas Sargent led to a revolution in macroeconomics by promoting an alternative approach called “rational expectations.”
Rational Expectations: Basic arguments

1. In modern macroeconomics, the term rational expectations means three things (*very important*):
   1. Agents use all publicly available information in an efficient manner, and therefore no systematic mistakes are produced when formulating expectations.
   2. Agents understand the structure of the model/economy and base their expectations of variables on this knowledge.
   3. Therefore, agents can forecast everything with no systematic mistakes; the only thing they cannot forecast are the *exogenous shocks* that hit the economy. These are unforecastable and unpredictable.

2. **Strong assumption**: the structure of the economy is complex and in truth nobody truly knows how everything works.

3. **How well does RE fit in reality?** Let’s look at some pictures
II – Some evidence on Rational Expectations
Evidence on Rational Expectations

Mistakes Forecasting the Recession of 1982

From N. G. Mankiw (2009), *Macroeconomics*, Worth Publishers
Evidence on Rational Expectations

Rational Expectations: How do they fit the facts

1. In the following figures we present some evidence about expectations of consumer prices and interest rates.

2. SPF - Survey of Professional Forecasters, Livingston Survey, and CPI (Consumer Price Index).

3. The Livingston Survey is the oldest continuous survey of economists’ expectations; it summarizes the forecasts of economists from industry, government, banking, and academia. The Survey of Professional Forecasters is the oldest quarterly survey of macroeconomic forecasts in the United States, began in 1968 and is managed by the Federal Reserve Bank of Philadelphia since 1990.

4. Data was obtained from the Federal Reserve Bank of Philadelphia website: http://www.philadephiafed.org/econ/forecast/

5. The Michigan survey on inflation expectations.
Survey expectations on inflation

![Graph showing survey expectations on inflation over time with lines for Livingston, SPF, and CPI.](image)
Evidence on Rational Expectations

Inflation: the most used Survey vs reality in the US
Biased expectations: CPI vs Michigan Survey on Inflation Expectations
Some points should be highlighted

1. Evidence seems to suggest that:
   1. For periods of high economic stability and low levels of inflation, expectations are very close to the true values of inflation.
   2. For periods of economic instability and high levels of inflation or very low levels of inflation, divergent expectations are easy to spot (systematic mistakes).

2. Do Shocks explain the divergence between expectations and true values? But what shocks?

3. So, probably what we may have is "Near Rational Expectations" (George Akerlof and Robert Shiller: in "Animal Spirits")

4. And sometimes even very irrational expectations.
III – Rational Expectations: the typical problem
Models with RE: the typical problem

1. Lots of models in economics take the form

\[ y_t = x_t + a \cdot E_t y_{t+1} \]  \hspace{1cm} (6)

2. It says that \( y \) today is determined by \( x \) today and by tomorrow’s expected value of \( y \). What determines this expected value?

3. Under RE, the agents understand what’s in the equation and formulate expectations in a way that is consistent with it:

\[ E_t y_{t+1} = E_t x_{t+1} + a \cdot E_t [E_{t+1} y_{t+2}] \]  \hspace{1cm} (7)

4. **We have a problem**: what is this term is the previous eq.?

\[ E_t [E_{t+1} y_{t+2}] \]

5. The **Law of Iterated Expectations** gives the answer.
Models with RE: the law of iterated expectations

1. The **Law of Iterated Expectations**: today, not rational to expect to have a different expectation at $t+1$ for $y_{t+2}$ than the one I have today.

2. Then, we get

$$ E_t [E_{t+1}y_{t+2}] = E_t y_{t+2} $$

3. And eq. (7) can be written as

$$ E_t y_{t+1} = E_t x_{t+1} + a \cdot E_t y_{t+2} $$ \tag{8}

4. Now we can substitute eq. (8) into equation (6), and get: **2nd iteration**

$$ y_t = x_t + a \cdot E_t x_{t+1} + a^2 E_t y_{t+2} $$

5. Let’s iterate forward once again by substituting for $E_t y_{t+2}$: **3rd iteration**

$$ y_t = x_t + a \cdot E_t x_{t+1} + a^2 E_t x_{t+2} + a^3 E_t y_{t+3} $$
Models with RE: the solution in compact form

1. At the 3rd iteration, we obtained this result

\[ y_t = x_t + a \cdot E_t x_{t+1} + a^2 E_t x_{t+2} + a^3 E_t y_{t+3} \]

2. This can be written in more useful (compact) form as

\[ y_t = \sum_{i=0}^{2} a^i E_t x_{t+i} + a^3 E_t y_{t+3} \]

Notice that when \( i = 0 \) we get \( a^i E_t x_{t+i} = E_t x_t = x_t \)

3. Generalizing for the \( n \)th iteration

\[ y_t = \sum_{i=0}^{n-1} a^i E_t x_{t+i} + a^n E_t y_{t+n} \]
Models with RE: a crucial assumption

1. We have just got this result

\[ y_t = \sum_{i=0}^{n-1} a^i E_t x_{t+i} + a^n E_t y_{t+n} \]  

(9)

2. Usually, it is assumed that \(|a| < 1\). This assumption makes sense as we will see later.

3. This assumption implies

\[ \lim_{n \to \infty} a^n E_t y_{t+n} = 0 \]

4. So the solution to equation (9) is

\[ y_t = \sum_{i=0}^{n-1} a^i E_t x_{t+i} \]

5. This solution underlies the logic of most modern macroeconomic models.
An example of RE: Asset Pricing

1. Consider a financial asset:
   1. Bought today at price $P_t$, and pays a dividend of $D_t$ per period.
   2. Assume a close substitute asset (e.g., a deposit with interest) that yields a safe rate of return given by $r$.

2. A risk neutral investor holds the asset if both assets get the same expected rate of return

\[
\frac{D_t + E_t P_{t+1}}{P_t} = 1 + r
\]

3. Solve for $P_t$, in order to simplify define $\phi = (1 + r)$, and get

\[
P_t = \frac{1}{\phi} D_t + \frac{1}{\phi} E_t P_{t+1}
\]
An example of RE: Asset Pricing (cont.)

1. By the usual method of repeated substitution, iterate forward

2. Take into account that, as $\phi = (1 + r) > 1$, then

   $$|1/\phi| < 1$$

3. And the final solution, at the $n$th iteration, is given by

   $$P_t = \sum_{i=0}^{n-1} (1/\phi)^{i+1} E_tD_{t+i}$$

4. **This is a very important result in finance**: asset prices should be equal to the discounted present-value sum of expected future dividends.
Solving Models with Rational Expectations
The usual problem in modern macroeconomics

1. Usually a macroeconomic model will have two types of variables:
   1. Predetermined or backward looking variables
   2. Forward looking variables

2. Firstly, we will look at the solution of a model involving each type of variable in a separate way

3. Then we will put both types of variables into a single model
IV – Solutions with predetermined variables
The solution with predetermined variables: forward iteration

1. Consider a model with **no forward looking expectations**.

2. Assume that $y_t$ is a state variable (or predetermined) and $x_t$ is an exogenous variable:

   \[ y_t = ay_{t-1} + x_t \]

3. Let’s find a solution by **forward iteration**:

   \[
   \begin{align*}
   y_1 &= a \cdot y_0 + x_1 \\
   y_2 &= a(ay_0 + x_1) + x_2 \\
   y_3 &= a[a(ay_0 + x_1) + x_2] + x_3 \\
   y_3 &= a^3 y_0 + a^2 x_1 + ax_2 + x_3 \\
   \ldots
   \end{align*}
   \]
Solution with predetermined variables: forward iteration

Forward iteration: the stability problem

1. At the \( n \) iteration, we get

\[
y_t = a^n y_0 + \sum_{i=0}^{n-1} a^i x_{t-i}
\]

2. \( y_t \) is a function of the past realizations of \( x \) and the initial condition \( y_0 \)

3. If \( |a| < 1 \): we get a stable solution because \( a^n y_0 \to 0 \), when \( n \to \infty \)
   
   - Acceptable solution, because \( y_t \) does not explode in finite time

4. If \( |a| > 1 \): there is no stable solution because \( a^n y_0 \to \infty \), when \( n \to \infty \)
   
   - Bad solution because \( y_t \) explodes in finite time
   - Counterintuitive: the weights on past values of \( x_t \) in the forward solution will explode
Summary

\[ y_t = a^n y_0 + \sum_{i=0}^{n-1} (a^i) x_{t-i} \] (11)

- If \( |a| < 1 \) : stable solution
- If \( |a| > 1 \) : no stable solution
- If \( |a| < 1 \) and \( x_t \) is not explosive, then \( y_t \) is also not explosive
  - The value of \( y_t \) depends only on the previous values of \( x_t \) (more on this later)
V – Forward looking variables: Backwards versus Forward solutions
Models with forward looking variables

1. Assume that $y_t$ is a \textbf{control variable} (like an interest rate chosen by the Central Bank, or public expenditure decided by the Government),

2. And $x_t$ is an exogenous economic variable

3. \textbf{Introduce forward looking expectations}, and for the sake of simplicity assume that the expectations term is on the left hand side

$$E_t y_{t+1} = ay_t + x_t$$

4. We may have two possible cases

$$|a| > 1 \quad ; \quad |a| < 1$$

5. Let’s see what happens when we try to get a solution
**Forward iteration**

1. Let’s try to get a solution by applying forward iteration to the eq.

\[ E_t y_{t+1} = a y_t + x_t \]

2. Rewrite and **iterate forward**

\[ y_t = \left( \frac{1}{a} \right) E_t y_{t+1} - \left( \frac{1}{a} \right) x_t \]

3. Get rid of the term \( E_t y_{t+i} \) by repeated substitution

\[ E_t y_{t+1} = \left( \frac{1}{a} \right) E_t y_{t+2} - \left( \frac{1}{a} \right) E_t x_{t+1} \]

\[ E_t y_{t+2} = \left( \frac{1}{a} \right) E_t y_{t+3} - \left( \frac{1}{a} \right) E_t x_{t+2} \]

\[ \vdots \]

4. And the result comes (at the \( n \)th iteration)

\[ y_t = \left( \frac{1}{a} \right)^n E_t y_{t+n} - \sum_{i=0}^{n-1} \left( \frac{1}{a} \right)^{i+1} E_t x_{t+i} \]
Forward iteration: the stability problem

\[ y_t = (1/a)^n E_t y_{t+n} - \sum_{i=0}^{n-1} (1/a)^{i+1} E_t x_{t+i} \]

1. If \(|a| > 1\): we get a stable solution because \((1/a)^n \to 0\), when \(n \to \infty\)
   - Acceptable solution, because \(y_t\) does not explode in finite time

2. If \(|a| < 1\): there is no stable solution because \((1/a)^n \to \infty\), when \(n \to \infty\)
   - Bad solution because \(y_t\) explodes in finite time
   - Counterintuitive: the weights on the expected future values of \(x_t\) in the forward solution will explode
Summary

\[ y_t = (1/a)^n E_t y_{t+n} - \sum_{i=0}^{n-1} (1/a)^{i+1} E_t x_{t+i} \]  
\( (12) \)

1. If \( |a| > 1 \) : stable solution
2. If \( |a| < 1 \) : no stable solution
3. If \( |a| > 1 \) : The value of \( y_t \) depends only on the \( E_t x_{t+i} \)

\[ y_t = - \sum_{i=0}^{n-1} (1/a)^{i+1} E_t x_{t+i} \]

1. If \( E_t x_{t+i} \) is stationary, then \( y_t \) will also be stationary
2. Later, we will see what happens if \( x_t \) is a stochastic process
Backward iteration

1. We know that if $|a| < 1$, the forward-iteration does not secure a stable solution.
2. Let’s try a **backward solution** as we did in the case of adaptive expectations

$$E_t y_{t+1} = ay_t + x_t$$ (13)

3. Define a new variable giving the value of the **forecasting error**

$$z_{t+1} = y_{t+1} - E_t y_{t+1}$$

4. Rewrite eq. (13) such that the process comes as

$$ay_t = y_{t+1} - z_{t+1} - x_t$$

5. Notice that the previous equation can be written as

$$y_t = ay_{t-1} + z_t + x_{t-1}$$
Backward iteration (continued)

1. The equation that we have to solve is
   \[ y_t = ay_{t-1} + z_t + x_{t-1} \]

2. Solving backwards in time, and considering that
   \[ y_{t-1} = ay_{t-2} + z_{t-1} + x_{t-2}; \quad y_{t-2} = ay_{t-3} + z_{t-2} + x_{t-3}; \quad y_{t-3} = \ldots \]

3. From the repeated substitution process, we get the following result at the 3rd iteration
   \[ y_t = a^3y_{t-3} + (a^2z_{t-2} + az_{t-1} + z_t) + (a^2x_{t-3} + ax_{t-2} + x_{t-1}) \]

4. Generalizing for the \( n \)th iteration

   \[ y_t = a^ny_{t-n} + \sum_{i=0}^{n-1} (a^i) z_{t-i} + \sum_{i=0}^{n-1} (a^i) x_{t-1-i} \]

5. Process is stable iff \( a^ny_{t-n} \) does not explode when \( n \to \infty \), that is iff \( |a| < 1 \).
Summary: backward vs Forward iteration with forward looking variables

1. **Both are correct solutions** to the first-order difference equation with rational expectations

   \[ E_t y_{t+1} = ay_t + x_t \]

2. Dependence on the **value of the parameter** \( a \).
   - \( |a| > 1 \) : choose **forward solution**
   - \( |a| < 1 \) : choose **backward solution**

3. Unfortunatelly, the backward solution will lead to a new major problem: **indeterminacy**
Indeterminacy

1. Imagine that the "fundamentals" of the economy $x_t$ are given. We can have as many solutions to $y_t$ as the different forecasting errors that may arise, due to the expectation term $\sum_{i=1}^{n-1} (a^i) z_{t-i}$ in the solution.

2. In this case we say that we have a continuum of solutions for any given future path for $x_t$. That’s indeterminacy, sometimes also called SUNSPOTS or ANIMAL SPIRITS.

3. The fundamentals of the economy $(x_t)$ are not the only factors that drive the final outcome: so, subjective beliefs about possible shocks may move the economy out of the path that is given by the fundamentals.

4. If $|a| > 1$, we adopt a forward iterating solution, and we get a stable value for $y_t$. NO INDETERMINACY! The value of $y_t$ will depend only on the economic fundamentals: the expected future path of $x_{t+i} : E_t x_{t+i}$.

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VI – The Blanchard-Kahn decoupling method
Methods to solve RE Models

1. There are a large number of techniques to solve a problem of a linear dynamic economic process with forward looking expectations.

2. The most used techniques are:
   1. Repeated substitution
   2. Method of undetermined coefficients
   3. The Blanchard-Kahn decoupling method
   4. The linear projection on observables

3. We will make use of the Blanchard-Kahn method throughout the remaining part of this course.

4. As most of the models in economics are multivariate, we revert now to matrices to deal with the problems of stability and indeterminacy in RE models.
Stability in Matrix form

1. **The main message** from our previous exercise is:
   1. Avoid explosive solutions,
   2. Even if you have a stable solution, be worried about multiple stable solutions (indeterminacy).

2. **Iterating forward**: to obtain a unique and stable solution we need
   1. $|a| > 1$ when dealing with a **forward looking** variable
   2. $|a| < 1$ when dealing with a **predetermined** variable

3. In the multivariate case, similar conditions apply to the eigenvalues of the following system

\[
\begin{align*}
A \cdot E_t z_{t+1} &= B \cdot z_t \\
E_t z_{t+1} &= A^{-1} \cdot B \cdot z_t \\
&= C
\end{align*}
\]
Stability in Matrix form (continued)

1. In matrix form

\[
A \cdot E_t z_{t+1} = B \cdot z_t \\
E_t z_{t+1} = A^{-1} \cdot B \cdot z_t = C
\]

2. If the number of eigenvalues \((\lambda_i)\) of \(C\) that lie outside the unit circle \((|\lambda_i| > 1)\)

   1. is equal to the number of forward-looking variables, there exists a unique and stable solution
   2. is larger than the number of forward-looking variables there is no stable solution
   3. is lower than the number of forward-looking variables there is an infinity of solutions
Some notes on Matrices and Eigenvalues

1. Assume we have a 2-dimensional system as follows

\[
x_{t+1} = ax_t + by_t \\
y_{t+1} = cx_t + dy_t
\]

where \(a, b, c, d\) are parameters.

2. This system can be written in matricial form as follows

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_{t+1} \\
y_{t+1}
\end{bmatrix}
= 
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\cdot
\begin{bmatrix}
x_t \\
y_t
\end{bmatrix}
\]

3. In matricial form we may write

\[
A \cdot \mathbf{z}_{t+1} = B \cdot \mathbf{z}_t \\
\mathbf{z}_{t+1} = A^{-1} \cdot B \cdot \mathbf{z}_t \\
\mathbf{z}_{t+1} = C \cdot \mathbf{z}_t
\]
How to obtain the eigenvalues

1. The eigenvalues of $C$ are values of $\lambda$ that satisfy the equation

$$[C - \lambda I] = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = 0$$

2. The determinant of $[C - \lambda I]$ is

$$\det [C - \lambda I] = (a - \lambda)(d - \lambda) - bc = 0$$

3. As matrix $C$ is of dimension 2, we get two possible values for lambda, $(\lambda_1, \lambda_2)$. **These are the eigenvalues for this system.**

4. If $|\lambda_1, \lambda_2| < 1$, the dynamics of our system is stable

5. If $|\lambda_1| < 1$ and $|\lambda_2| > 1$, we have saddle path stability

6. If $|\lambda_1, \lambda_2| > 1$, we get explosive behavior
We wrote our RE system as

\[ E_t z_{t+1} = A^{-1} \cdot B \cdot z_t = C \cdot z_t \] (14)

Example: take a system of dimension 3, for example, \( z_t \) contains:

1. 2 predetermined (or state) variables
2. 1 control (or forward looking) variable

Iterate the system forwards and avoid explosive behavior

From the univariate case stability implies:

1. \(| a | > 1\) when dealing with a forward looking variable
2. \(| a | < 1\) when dealing with a predetermined variable

By analogy: in this multivariate example, a unique and stable solution requires:

1. \( | \lambda_1 | > 1 \), for the forward looking variable
2. \( | \lambda_2, \lambda_3 | < 1 \), for the 2 predetermined variables

This is what we mean when we want a stable and unique solution to RE models.
The proof that comes next is not required for this course
The basic idea behind the B-K method

1. Apply the Jordan decomposition to transform our complicated models into two distinct blocks:
   - The block including only predetermined variables \(x_t\)
   - The other block including only forward looking variables \(E_t y_{t+1}\)

2. Then we can apply the following strategy:
   - Iterate forward
   - Firstly, the predetermined block
   - Then, the forward looking block.

3. And we get this type of results

\[
\text{Shocks: } v(t) \quad x(t) \quad y(t) \\
x(t - 1)
\]

The Jordan decomposition

1. Compute the Jordan canonical form (also called Jordan normal form) of a symbolic or numeric matrix \( A \).

2. Our model comprises: a set of predetermined variables \((x_t)\), a set of forward looking variables \((y_t)\), and a set of exogenous shocks \((v_t)\).

3. Write the model in state space form

\[
\begin{bmatrix}
    w_{t+1} \\
    E_t y_{t+1}
\end{bmatrix} = A \begin{bmatrix}
    w_t \\
    y_t
\end{bmatrix} + B v_{t+1}
\]

(15)

4. The Jordan decomposition of \((A)\)

\[
A = P \Lambda P^{-1}
\]

5. \( \Lambda \) is a diagonal matrix with the \textbf{eigenvalues} of \( A \) along its leading diagonal and zeros in the remaining entries.

6. \( P \) contains the inverse matrix of the generalized \textbf{eigenvectors} of \( A \) as columns.
The model with the Jordan decomposition

1. Apply the decomposition

\[
\begin{bmatrix}
w_{t+1} \\
E_t y_{t+1}
\end{bmatrix} = P \Lambda P^{-1} \begin{bmatrix}
w_t \\
y_t
\end{bmatrix} + B v_{t+1}
\] (16)

2. Multiply both sides by \( P^{-1} \)

\[
P^{-1} \begin{bmatrix}
w_{t+1} \\
E_t y_{t+1}
\end{bmatrix} = \Lambda P^{-1} \begin{bmatrix}
w_t \\
y_t
\end{bmatrix} + P^{-1} B \cdot v_{t+1}
\] (17)
The model with the Jordan decomposition

1. Partition $P^{-1}$ and $\Lambda$ to get

$$
\begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix}
\begin{bmatrix}
w_{t+1} \\
E_t y_{t+1}
\end{bmatrix}
= 
\begin{bmatrix}
\Lambda_1 & 0 \\
0 & \Lambda_2
\end{bmatrix}
\begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix}
\begin{bmatrix}
w_t \\
y_t
\end{bmatrix}
+ 
\begin{bmatrix}
\tilde{\omega}_{t+1} \\
E_t \tilde{y}_{t+1}
\end{bmatrix}
$$

(18)

2. So our transformed model looks much easier now

$$
\begin{bmatrix}
\tilde{\omega}_{t+1} \\
E_t \tilde{y}_{t+1}
\end{bmatrix}
= 
\begin{bmatrix}
\Lambda_1 & 0 \\
0 & \Lambda_2
\end{bmatrix}
\begin{bmatrix}
\tilde{\omega}_t \\
\tilde{y}_t
\end{bmatrix}
+ 
\begin{bmatrix}
R_1 \\
R_2
\end{bmatrix}
\nu_{t+1}
$$
The two decoupled blocks

Transformed model written down as a set of decoupled equations

\[
\begin{align*}
\tilde{w}_{t+1} &= \Lambda_1 \tilde{w}_t + R_1 v_{t+1} \\
E_t \tilde{y}_{t+1} &= \Lambda_2 \tilde{y}_t + R_2 v_{t+1}
\end{align*}
\]

(Stable block)

(Unstable block)

1. We can now apply our old strategy
   1. Solve the unstable transformed block forward and get: \( \tilde{y}_t^* \)
   2. Solve the stable transformed block backwards and get: \( \tilde{w}_t^* \)

2. Insert the results back into the original problem
Solving the unstable block

1. Iterating forward this block, we get
   \[ E_t \tilde{y}_{t+n} = (\Lambda_2)^n \tilde{y}_t \]

2. If we have
   \[ |\Lambda_2| > 1 \]

3. Then, the only stable solution will be
   \[ \tilde{y}_t^* = 0, \forall t \]

4. Now from our definition in eq. (18), we know that
   \[ \tilde{y}_t^* = P_{21} \cdot w_t^* + P_{22} \cdot y_t^* = 0 \]

5. From which
   \[ y_t^* = \left[ -P_{22}^{-1} P_{21} \right] \cdot w_t^* \] \hspace{1cm} (19)

6. **Notice that this our old result:** forward looking variables depending upon predetermined ones.
Solving the stable block

1. Iterating forward this block, we get

\[ \tilde{w}_{t+n} = (\Lambda_1)^n \tilde{w}_t, \quad |\Lambda_1| < 1 \]

2. If we assume that

\[ |\Lambda_1| < 1 \]

3. The process is stable, and from eq. (18), we get

\[ \tilde{w}_t^* = P_{11} \cdot w_t^* + P_{12} \cdot y_t^* \quad (20) \]

4. Now insert eq. (19) into (20), and get

\[ \tilde{w}_t^* = \left[ P_{11} - P_{12}P_{22}^{-1}P_{21} \right] \cdot w_t^* \quad (21) \]
Solving the stable block (cont.)

1. But as from eq. (Stable block), we have

\[ \tilde{w}_{t+1} = \Lambda_1 \tilde{w}_t + R_1 v_{t+1} \]

2. And as from eq. (21) we have

\[ \tilde{w}_t^* = D \cdot w_t^* \]

3. Then

\[ D \cdot w_{t+1}^* = \tilde{w}_{t+1}^* \]

\[ \tilde{w}_t^* = D \cdot w_t^* \]

\[ \tilde{w}_{t+1} = \Lambda_1 \tilde{w}_t + R_1 v_{t+1} \]

4. From which we finally get

\[ w_{t+1}^* = \left[ D^{-1} \Lambda_1 D \right] w_t^* + \left[ D^{-1} R_1 \right] v_{t+1} \]  

(22)
Summarizing

1. Write down your model in state space form
2. Apply the Jordan decomposition
3. Decouple the system into two blocks
4. Make sure one eigenvalue is larger than 1 in modulus, the other lower than 1 in modulus.
5. End up with the two fundamental results

\[ y_t^* = \left( -P_{22}^{-1}P_{21} \right) \cdot w_t^* \]

\[ w_{t+1}^* = \left( D^{-1} \Lambda_1 D \right) w_t^* + \left( D^{-1} R_1 \right) v_{t+1} \]

with \( D = P_{11} - P_{12} (P_{22})^{-1} P_{21} \)
V – Bibliography
Fabrice Collard (2015). "Expectations and Economic Dynamics", Lecture Notes, University of Bern. Read only sections 1.2.1, 1.2.2 (here do not read the parts dealing with "Factorization" and "The Method of Indeterminate Coefficients"), and section 1.2.3.

David Peel (2005). "Advanced Macroeconomics" (Chapter 2), unpublished manuscript, Lancaster University, UK.

It's a long text that can be used as complementary studying material, but never as compulsory reading.