A Two Period Economy

— Week 5 —

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Summary

1. Intertemporal economic decision making
2. The Lagrange Method
3. Liquidity constraints
4. Endogenous labor supply
5. Bibliography

I – Intertemporal economic decision making

Very similar to static optimization decision making

1. Agents try to do as well as possible, with the available information
2. Agents are subjected to constraints (financial or technological)
3. Agents **are rational** (even if some form of limited rationality)
4. They face two types of variables: **state** (stocks) and **control** (flows)

The differences

The differences are that in intertemporal decision making:

1. The decisions are made over time, and not only at one particular point in time
2. Be aware of a common mistake that may arise: mix up long term equilibria with short term transitional dynamics
3. Allows us to analyze the sustainability of economic processes
4. Allows us to analyze the **intertemporal consistency** of the decisions that are made over time
5. Uses specific mathematical techniques: difference equations, differential equations or partial deferential equations

The typical household problem: preferences

1. Agents have preferences

$$U(c_t, c_{t+1}) = u(c_t) + \frac{1}{1+R} \cdot u(c_{t+1})$$

2. Or in a more simple formulation

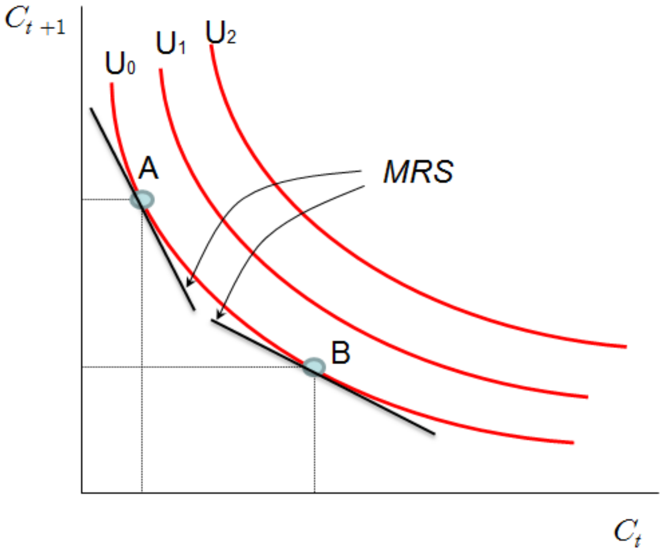
$$U(c_t, c_{t+1}) = u(c_t) + \beta \cdot u(c_{t+1})$$

$u-$ utility index, $c-$ consumption levels
$R-$ subjective rate ate of discounting utility
$\beta \in (0,1)-$ factor or gross rate of intertemporal discounting of utility

3. Important concept: marginal rate of substitution between c_t and c_{t+1}

$$MRS_{t,t+1} = -\frac{\partial U / \partial c_t}{\partial U / \partial c_{t+1}} = -\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})}$$

The typical household problem: preferences



The typical household problem: constraints

1. The constraints for the two periods of time are

$$\begin{aligned} c_t + a_{t+1} &\leq w_t \\ c_{t+1} &\leq (1 + r_{t+1})a_{t+1} + w_{t+1} \end{aligned}$$

   $a_{t+1}-$ savings at t and transformed into a financial asset at $t+1$
   w_t- wages or income at t , $r_{t+1}-$ interest rate (return on financial assets)

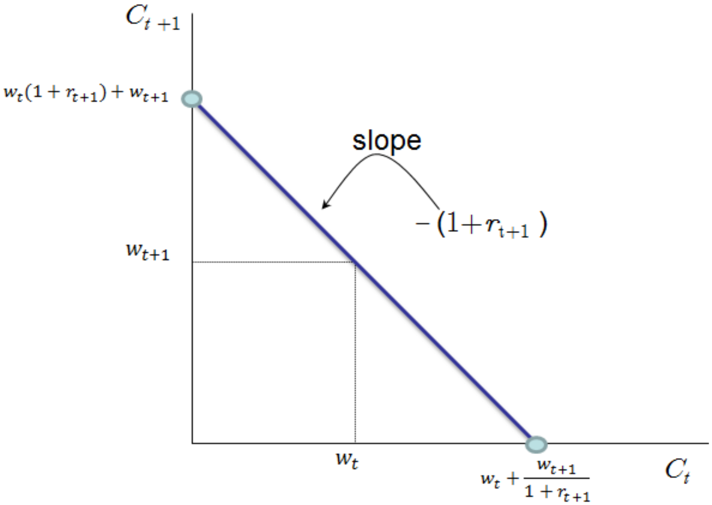
2. The consolidated intertemporal constraint: eliminate a_{t+1}. **Expressed in current values**

$$\underbrace{c_t + \frac{c_{t+1}}{1 + r_{t+1}}}_{\text{value of intertemporal consumption}} \leq \underbrace{w_t + \frac{w_{t+1}}{1 + r_{t+1}}}_{\text{value of intertemporal income}}$$

3. The consolidated intertemporal constraint: eliminate a_{t+1}. **Expressed in future values**

$$(1 + r_{t+1})c_t + c_{t+1} \leq (1 + r_{t+1})w_t + w_{t+1}$$

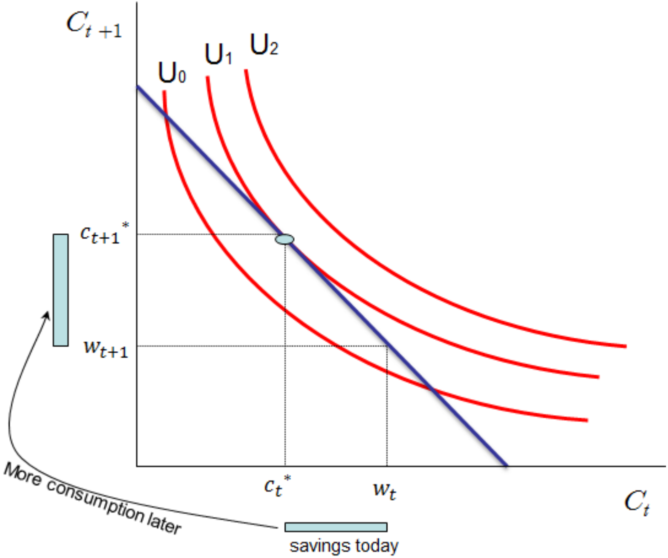
The typical household problem: constraints



The maximization of utility

1. It implies the finding of the optimal levels of consumption (c_t^*, c_{t+1}^*) and of savings (a_{t+1}^*) over time
2. This can be done in two ways: graphically, algebraically
3. The slope of the intertemporal budget constraint is given by
$$-(1+r_{t+1})$$
4. The Marginal Rate of Substitution is given by
$$MRS_{t,t+1} = -\frac{\partial U/\partial c_t}{\partial U/\partial c_{t+1}} = -\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})}$$
5. Equalizing both, we get
$$\begin{aligned} -\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})} &= -(1+r_{t+1}) \\ u'(c_t) &= (1+r_{t+1}) \cdot \beta \cdot u'(c_{t+1}) \end{aligned}$$

The typical household problem: constraints



II – The Lagrange Method

What is it?

1. In everything similar to the Lagrangean function that you learned in every intermediate course in Microeconomics
2. Very useful if time is discrete
3. Very intuitive
4. Highly used in modern macroeconomics everywhere
5. We have:
    1. An objective function
    2. A constraint (or various constraints)
    3. Viability conditions (to assure the optimal values)

Formally speaking

1. Objective: to maximize utility (now, intertemporal)

$$\max_{c_t,\, c_{t+1},\, a_{t+1}} u(c_t) + \beta \cdot u(c_{t+1})$$

2. There are two constraints:

$$c_t + a_{t+1} \leq w_t$$
$$c_{t+1} \leq (1 + r_{t+1})a_{t+1} + w_{t+1}$$

3. Viability conditions are: $c_t, c_{t+1} \geq 0$
4. Formal setting of the problem

$$\max_{c_t,\, c_{t+1},\, a_{t+1}} u(c_t) + \beta \cdot u(c_{t+1})$$

$$\text{s.t.} \quad c_t + a_{t+1} \leq w_t$$
$$c_{t+1} \leq (1 + r_{t+1})a_{t+1} + w_{t+1}$$
$$c_t,\; c_{t+1} \geq 0$$

The Lagrangean

1. The Lagrangean function is written as

$$\begin{aligned} \mathcal{L} &= u(c_t) + \beta \cdot u(c_{t+1}) + \\ & \quad \lambda_t(w_t - c_t - a_{t+1}) + \lambda_{t+1}[w_{t+1} + (1 + r_{t+1})a_{t+1} - c_{t+1}] \\ c_t,\ c_{t+1} &\geq 0, \\ \lambda_t,\ \lambda_{t+1} &\geq 0 \end{aligned}$$

2. How many unknowns?
3. Five unknowns: $c_t,\ c_{t+1}, a_{t+1}, \lambda_t,\ \lambda_{t+1}$
4. Five First Order Conditions **(FOC)**

First Order Conditions

1. The 5 FOCs are

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow u'(c_t) - \lambda_t = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = 0 \Rightarrow \beta \cdot u'(c_{t+1}) - \lambda_{t+1} = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = 0 \Rightarrow -\lambda_t + \lambda_{t+1}(1 + r_{t+1}) = 0 \quad (3)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \Rightarrow w_t - c_t - a_{t+1} = 0 \quad (4)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_{t+1}} = 0 \Rightarrow w_{t+1} + (1 + r_{t+1})a_{t+1} - c_{t+1} = 0 \quad (5)$$

2. Now there is always a very useful trick: **you can eliminate the $\lambda's$.**
3. Then simplify
4. Let's do it.

Simplification

1. From the first two FOCs (FOC1, FOC2)
$$\begin{aligned}\partial\mathcal{L}/\partial c_t &= 0 \Rightarrow u'(c_t) - \lambda_t = 0\\ \partial\mathcal{L}/\partial c_{t+1} &= 0 \Rightarrow \beta \cdot u'(c_{t+1}) - \lambda_{t+1} = 0\end{aligned}$$
2. we get
$$\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})} = \frac{\lambda_t}{\lambda_{t+1}} \tag{6}$$
3. From FOC3 $-\lambda_t + \lambda_{t+1}(1 + r_{t+1}) = 0$ we get
$$\lambda_t/\lambda_{t+1} = 1 + r_{t+1} \tag{7}$$
4. Equalizing the two previous eq., we get our already known result
$$\underbrace{\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})}}_{MRS_{t,t+1}} = \underbrace{(1 + r_{t+1})}_{relative\ price}$$
5. So, either graphically, or by the Lagrange method we arrived at the

The Euler Equation

1. We have just got this result

$$\underbrace{\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})}}_{MRS_{t,t+1}} = \underbrace{(1 + r_{t+1})}_{relative\ price}$$

2. In an equivalent way, known as the **EULER EQUATION**

$$u'(c_t) = (1 + r_{t+1}) \cdot \beta \cdot u'(c_{t+1})$$

3. The Euler equation is extremely important in modern macroeconomics:
    1. allows to incorporate microfountations into macroeconomics (preferences of the demand side)
    2. allows to measure the welfare of different macroeconomic policies
4. Three possible cases can be easily spotted.

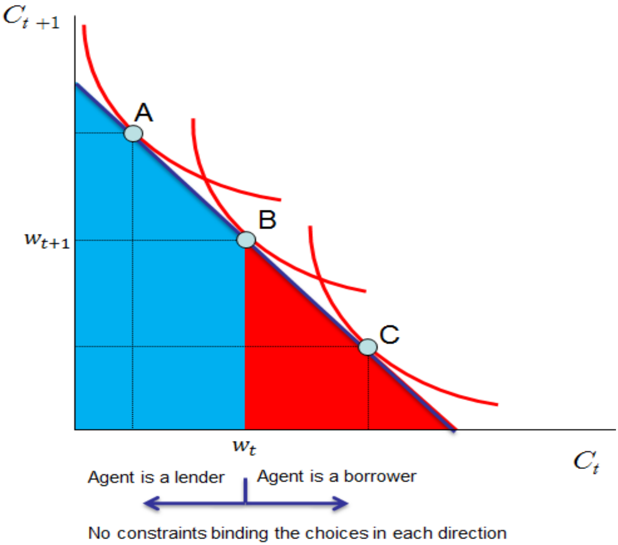
$$\begin{aligned}(1 + r_{t+1}) \cdot \beta &= 1 \Rightarrow u'(c_t) = u'(c_{t+1}) \\ (1 + r_{t+1}) \cdot \beta &\lesseqgtr 1 \Rightarrow u'(c_t) \lesseqgtr u'(c_{t+1})\end{aligned}$$

III – Liquidity constraints

The liquidity constraint problem

1. Up to now, we have assumed that the agents face **no debt constraint**
2. **Any transaction** between current and future consumption is possible.
3. Whenever one agent wants to increase current consumption, there is always some other agent that wants the opposite.
4. If there is a mismatch between the savings of some and the debt intentions of others, the relative price of intertemporal consumption will change $(1 + r_{t+1})$, so that equilibrium will be reestablished again
5. That is, up to now: **MARKETS WERE COMPLETE**
6. What happens if markets are not complete? If some agents are faced with debt constraints?

The no-liquidity constraint case



Binding versus Non-Binding constraints

1. Assume that if a household has negative savings

$$a_{t+1}^{-}$$

2. ... **can not borrow more than** a certain value at t

$$a_{t+1}^{-} \leq A$$

3. Solution: totally equal to the solution with complete markets
4. The only difference now is that these constraints should apply

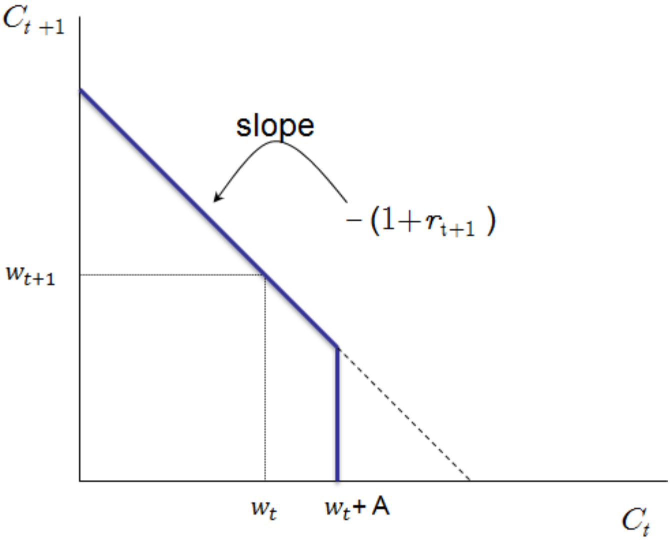
$$\begin{aligned} a_{t+1}^{-} &\leq A \\ c_t,\ c_{t+1} &\geq 0 \end{aligned}$$

5. No Ponzii game: at $t+1$ the borrower must be able to repay debt

$$(1 + r_{t+1})\, A < w_{t+1}$$

6. Binding versus Non-Binding constraints:

$$\begin{aligned} a_{t+1}^{-} &= A \quad , \quad \text{Binding} \\ a_{t+1}^{-} &< A \quad , \quad \text{Non-Binding} \end{aligned}$$


C_{t+1}
slope
$-(1+r_{t+1})$
w_{t+1}
w_t
w_t+ A
C_t

A binding constraint

1. This occurs if

$$a_{t+1}^{-} = A$$

2. The agent wants to consume more at t, but he cannot do it because he is financially constrained

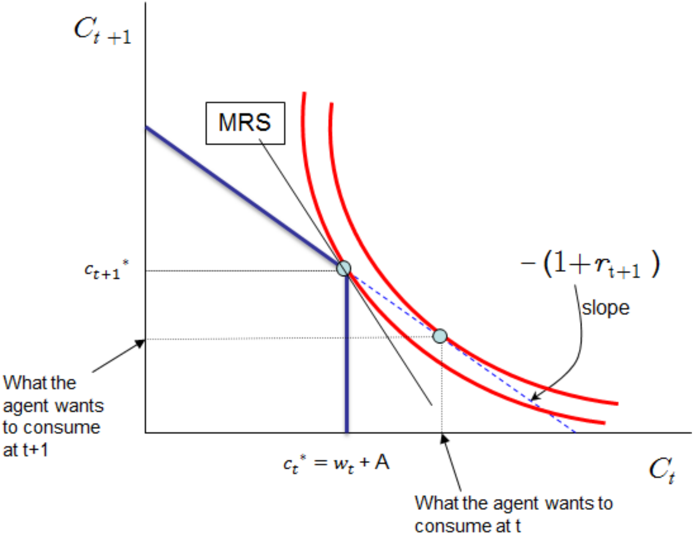
$$MRS > relative\ price$$

3. Mathematically:

$$\underbrace{\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})}}_{MRS_{t,t+1}} > \underbrace{(1 + r_{t+1})}_{relative\ price}$$

4. **Implications**? **Economic inefficiency**: the welfare of our constrained agent is lower than without the binding constraint.

An example of a binding constrain



A non–binding constraint

1. This occurs if

$$a^{-}_{t+1} < A$$

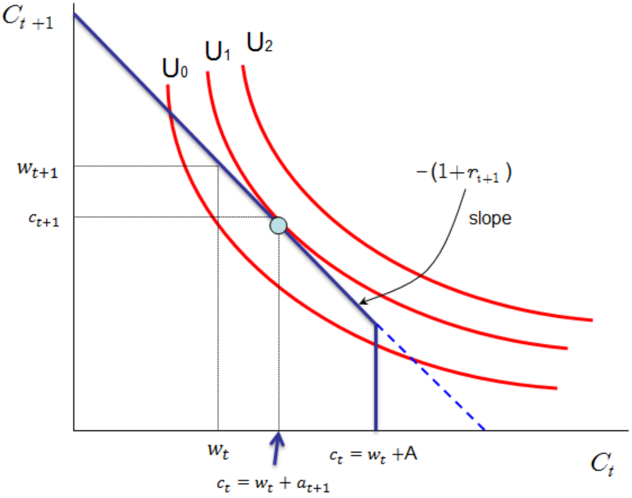
2. Solution totally equal to the solution without debt constraints
3. No economic inefficiency

$$MRS = relative\ price$$

4. Mathematically:

$$\underbrace{\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})}}_{MRS_{t,t+1}} = \underbrace{(1 + r_{t+1})}_{relative\ price}$$

A non–binding constraint example



IV – Endogenous labor supply

The two fundamental assumptions: up to now

1. The results obtained depended on two fundamental assumptions.
2. **Assumption1**. The level of income in both periods was taken as exogenous

$$w_t, w_{t+1} \text{ were given}$$

3. **Assumption 2**. The level of leisure (and the labor effort) was taken as exogenous

$$\begin{aligned} \ell &= \bar{\ell} \text{ , labor effort as given} \\ & 1-\ell, \text{ leisure time given} \end{aligned}$$

4. However, in reality we know that the two ingredients in the previous assumptions are linked

wages can increase, if the labor effort increases

5. So what happens to intertemporal utility?

The new intertemporal problem

1. How can this link be integrated into the model? Through the utility function.
2. Working more hours
    1. leads to higher income, which leads to higher consumption, which implies higher utility levels
    2. However, it also implies less leisure time, which implies lower utility
    3. There is a trade-off here
3. So now we will have two intertemporal decisions, not just one:
    1. Not only, should I consume more now, or more in the future?
    2. But also, should I work more hours, or should I enjoy more leisure?

$$\uparrow \quad \ell \Rightarrow \uparrow \mathit{income} \Rightarrow \uparrow \mathit{consumption} \Rightarrow \uparrow \mathit{utility}$$

$$\uparrow \quad \ell \Rightarrow \downarrow \mathit{leisure} \Rightarrow \downarrow \mathit{utility}$$

Setting the new intertemporal problem

1. Symbols:
   ℓ : labor effort; $1-\ell$: leisure
   β : intertemporal discount factor
2. Agents preferences

$$\begin{aligned} U(c,\ell) &= u(c_t, 1-\ell_t) + \beta \cdot u(c_{t+1}, 1-\ell_{t+1}) \\ & \quad \textit{or} \text{ in another way} \\ U(c,\ell) &= u(c_t, \ell_t) + \beta \cdot u(c_{t+1}, \ell_{t+1}) \end{aligned}$$

3. Let's impose some useful conditions on the utility function

$$\begin{pmatrix} u'_c > 0 & , & u''_c < 0 \\ u'_{1-\ell} > 0 & , & u''_{1-\ell} < 0 \\ u'_\ell < 0 & , & u''_\ell > 0 \end{pmatrix}$$

Setting the new intertemporal problem (cont.)

1. Intertemporal constraints

$$c_t + a_{t+1} \leq w_t \ell_t$$
$$c_{t+1} \leq (1 + r_{t+1}) a_{t+1} + w_{t+1} \ell_{t+1}$$

2. Consolidated constraint: eliminate a_{t+1}. **Expressed in current values**

$$\underbrace{c_t + \frac{c_{t+1}}{1 + r_{t+1}}}_{\textit{value of intertemporal consumption}} \quad \leq \quad \underbrace{w_t \ell_t + \frac{w_{t+1} \cdot \ell_{t+1}}{1 + r_{t+1}}}_{\textit{value of intertemporal income}}$$

3. Maximizing utility

$$\max_{c_t, c_{t+1}, \ell_t, \ell_{t+1}, a_{t+1}} u(c_t, 1 - \ell_t) + \beta \cdot u(c_{t+1}, 1 - \ell_{t+1})$$

$$\text{s.t.} \quad c_t + a_{t+1} \leq w_t \ell_t$$
$$c_{t+1} \leq (1 + r_{t+1}) a_{t+1} + w_{t+1} \ell_{t+1}$$
$$c_t, \ c_{t+1} \geq 0, \quad \ell_t, \ \ell_{t+1} \in (0, 1)$$

The Lagrangean function

1. The Lagrangean function is written as

$$\begin{aligned}\mathcal{L} &= u(c_t, 1-\ell_t) + \beta \cdot u(c_{t+1}, 1-\ell_{t+1}) + \lambda_t(w_t\ell_t - c_t - a_{t+1}) + \\ &\quad \lambda_{t+1}[w_{t+1}\ell_{t+1} + (1+r_{t+1})a_{t+1} - c_{t+1}]\end{aligned}$$

$$c_t,\ c_{t+1} \geq 0, \quad \ell_t,\ \ell_{t+1} \in (0,1), \quad \lambda_t,\ \lambda_{t+1} \geq 0$$

2. λ_t are the Lagrange multipliers
3. How many unknowns? Seven.

$$c_t, \quad c_{t+1}, \quad \ell_t, \quad \ell_{t+1}, \quad a_{t+1}, \quad \lambda_t, \quad \lambda_{t+1}$$

4. How many First Order Conditions? Seven

The First Order Conditions (FOCs)

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= 0 \Rightarrow u'_{c_t} - \lambda_t = 0 \\
\frac{\partial \mathcal{L}}{\partial c_{t+1}} &= 0 \Rightarrow \beta \cdot u'_{c_{t+1}} - \lambda_{t+1} = 0 \\
\frac{\partial \mathcal{L}}{\partial \ell_t} &= 0 \Rightarrow -u'_{1-\ell_t} - \lambda_t w_t = 0 \\
\frac{\partial \mathcal{L}}{\partial \ell_{t+1}} &= 0 \Rightarrow -\beta \cdot u'_{1-\ell_{t+1}} + \lambda_{t+1} w_{t+1} = 0 \\
\frac{\partial \mathcal{L}}{\partial a_{t+1}} &= 0 \Rightarrow -\lambda_t + \lambda_{t+1}(1 + r_{t+1}) = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_t} &= 0 \Rightarrow w_t \ell_t - c_t - a_{t+1} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_{t+1}} &= 0 \Rightarrow w_{t+1} \ell_{t+1} + (1 + r_{t+1}) a_{t+1} - c_{t+1} = 0
\end{aligned}
$$

The Euler Equation

1. From the first two FOCs

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow u'_{c_t} - \lambda_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = 0 \Rightarrow \beta \cdot u'_{c_{t+1}} - \lambda_{t+1} = 0$$

2. we get

$$\frac{u'_{c_t}}{\beta \cdot u'_{c_{t+1}}} = \frac{\lambda_t}{\lambda_{t+1}}$$

3. From FOC5 $-\lambda_t + \lambda_{t+1}(1 + r_{t+1}) = 0$ we get

$$\frac{\lambda_t}{\lambda_{t+1}} = 1 + r_{t+1}$$

4. Equalizing the two previous eq., we get our already known **Euler Equation**

$$u'_{c_t} = (1 + r_{t+1}) \cdot \beta \cdot u'_{c_{t+1}} \tag{8}$$

Optimal intertemporal labor effort

1. Use FOC3, FOC4, and combine, respectively, with FOC1 and FOC2
2. We will get

$$\frac{u'_{1-\ell_t}}{u'_{c_t}} = w_t$$

$$\frac{u'_{1-\ell_{t+1}}}{u'_{c_{t+1}}} = w_{t+1}1$$

3. Simplifying, we get a static condition for each period that optimizes the labor effort

$$\underbrace{u'_{c_t}}_{\text{marginal utility of } c} = \underbrace{\frac{u'_{1-\ell_t}}{w_t}}_{\text{marginal utility of leisure weighted by its price}}$$

$$\underbrace{u'_{c_{t+1}}}_{\text{marginal utility of } c} = \underbrace{\frac{u'_{1-\ell_{t+1}}}{w_{t+1}}}_{\text{marginal utility of leisure weighted by its price}}$$

Summary: the tree intertemporal optimal conditions

1. One dynamic condition: the Euler equation, should I consume more now, or should I consume more in the future?

$$u'_{c_t} = (1 + r_{t+1}) \cdot \beta \cdot u'_{c_{t+1}}$$

2. Two static conditions: should I work more, or should I work less?

$$u'_{c_t} = \frac{u'_{1-\ell_t}}{w_t}$$

$$u'_{c_{t+1}} = \frac{u'_{1-\ell_{t+1}}}{w_{t+1}}$$

Example 1

1. Assume that agents only work in the first period
2. Assume that the utility function is logarithmic

$$U(c,\ell) = \ln c_t + \ln(1-\ell_t) + \beta \ln c_{t+1}$$

3. Utility maximization

$$\max_{c_t, c_{t+1}, \ell_t, a_{t+1}} \ln c_t + \ln(1-\ell_t) + \beta \ln c_{t+1}$$

$$\text{s.t..} \quad c_t + a_{t+1} \leq w_t \ell_t$$
$$c_{t+1} \leq (1+r_{t+1})a_{t+1} + w_{t+1} \cdot \underbrace{\ell_{t+1}}_{0}$$

$$c_t,\ c_{t+1} \geq 0, \quad \ell_t \in (0,1)$$

Example 1 (cont.)

1. Applying directly the Euler equation

$$u'_{c_t} = (1 + r_{t+1}) \cdot \beta \cdot u'_{c_{t+1}}$$

2. We get

$$\frac{1}{c_t} = (1 + r_{t+1}) \cdot \beta \cdot \frac{1}{c_{t+1}} \tag{9}$$

3. That is

$$c_{t+1} = \beta \left(1 + r_{t+1}\right) c_t$$

4. Now applying the first static condition $u'_{c_t} \cdot w_t = u'_{1-\ell_t}$, we get

$$\begin{aligned} \frac{1}{c_t} \cdot w_t &= \frac{1}{1 - \ell_t} \\ c_t &= w_t \left(1 - \ell_t\right) \end{aligned} \tag{10}$$

Example 1 (cont.)

1. Using the intertemporal consolidated constraint (notice that $\ell_{t+1} = 0$)

$$c_t + \frac{c_{t+1}}{1 + r_{t+1}} \leq w_t \ell_t$$

2. ...together with the Euler Equation (9), we get

$$c_t = \frac{1}{1+\beta} w_t \ell_t \tag{11}$$

3. Now we can determine the optimal intertemporal value of labor effort
4. Just combine eq(10) and eq(11) and we get

$$\ell_t^* = \frac{1+\beta}{2+\beta}$$

Example 1: final results

1. We have just got the optimal level of labor effort for period t

$$\ell_t^* = \frac{1+\beta}{2+\beta}$$

2. Applying back into eq(11), we get

$$c_t^* = \frac{1}{2+\beta} w_t$$

3. And once we know c_t^*, we can easily determine c_{t+1}^*

$$c_{t+1}^* = \left(\frac{w_t}{2+\beta} \right) \beta(1 + r_{t+1})$$

4. **Important conclusion: the supply of labor is inelastic with respect to wages.** (ℓ_t^* is independent from w_t)

Example 2

1. Assume now that agents have an initial level of wealth $a_t > 0$.
2. Moreover, they continue working only in period t
3. We will see that the conclusion in the previous slide is totally reversed.
4. The problem now looks like

$$\max_{c_t, c_{t+1}, \ell_t, a_{t+1}} \ln c_t + \ln(1-\ell_t) + \beta \ln c_{t+1}$$

$$\text{s.t.} \quad c_t + a_{t+1} \leq w_t \ell_t + a_t(1+r_t)$$
$$c_{t+1} \leq (1+r_{t+1})a_{t+1} + w_{t+1} \cdot \underbrace{\ell_{t+1}}_{0}$$

$$c_t, \; c_{t+1} \geq 0, \quad \ell_t \in (0,1)$$

Example 2 (cont.)

1. From the two FOCs with respect to ℓ_t, ℓ_{t+1}, we will get

$$w_t\left(1-\ell_t\right)(1+\beta)=w_t\ell_t+a_t(1+r_t)$$

2. From where we can obtain the optimal level of labor effort

$$\ell_t^*=\frac{1}{2+\beta}\left[1+\beta-\frac{a_t(1+r_t)}{w_t}\right]$$

3. Therefore, in this case we have

$$\uparrow w_t \Rightarrow \uparrow \ell_t^* \quad , \quad \uparrow r_t \Rightarrow \downarrow \ell_t^* \quad , \quad \uparrow a_t \Rightarrow \downarrow \ell_t^*$$

V – Bibliography

Bibliography

If you are able to read Spanish, the next reading is very,very good:

J.C. Conesa and C. Garriga (2011). *Teoria Económica del Capital y la Renta*, Universitat Autònoma de Barcelona. *Read chapter 4 (for points 1, 2 and 3) and Chapter 5 (for point 5). Follow the slides, as some parts of each chapter are skipped without loss of any relevant information for the course.*

If you prefer a text in English, the next one is concise but very good:

Martin Boileau (2001). "Two Period Economies: A Review", University of Colorado, Boulder. *For those of are not fluent in reading in Spanish, this is a good alternative to the reference above.*

Bibliography (cont.)

Another short text in English, concise, good, and with too many questions along the text (do not worry about these questions) is:

S. Shi (2009). "Two Period Economies", University of Toronto, Toronto.

If you have not dealt with any kind of this stuff before in your studies and you feel a little bit uneasy, *for an introductory treatment of the issues discussed in this week. See Chapters 4 and 8, at the expenses of having to read a much larger number of pages.*

Stephen Williamson (2011). *Macroeconomics*, Pearson, New York.