

# A Two Period Economy

— Week 5 —

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# Summary

- 1 Intertemporal economic decision making
- 2 The Lagrange Method
- 3 Liquidity constraints
- 4 Endogenous labor supply
- 5 Bibliography

# I – Intertemporal economic decision making

# Very similar to static optimization decision making

- 1 Agents try to do as well as possible, with the available information
- 2 Agents are subjected to constraints (financial or technological)
- 3 Agents **are rational** (even if some form of limited rationality)
- 4 They face two types of variables: **state** (stocks) and **control** (flows)

# The differences

The differences are that in intertemporal decision making:

- 1 The decisions are made over time, and not only at one particular point in time
- 2 Be aware of a common mistake that may arise: mix up long term equilibria with short term transitional dynamics
- 3 Allows us to analyze the sustainability of economic processes
- 4 Allows us to analyze the **intertemporal consistency** of the decisions that are made over time
- 5 Uses specific mathematical techniques: difference equations, differential equations or partial differential equations

# The typical household problem: preferences

- 1 Agents have preferences

$$U(c_t, c_{t+1}) = u(c_t) + \frac{1}{1+R} \cdot u(c_{t+1})$$

- 2 Or in a more simple formulation

$$U(c_t, c_{t+1}) = u(c_t) + \beta \cdot u(c_{t+1})$$

$u$  – utility index,  $c$  – consumption levels

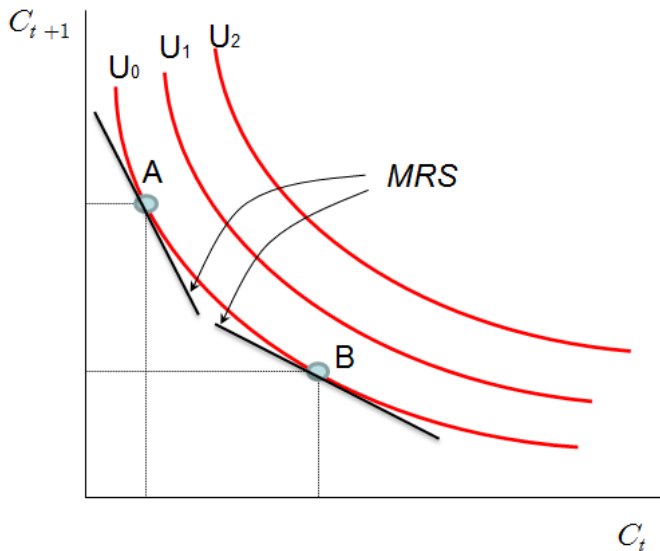
$R$  – subjective rate of discounting utility

$\beta \in (0, 1)$  – factor or gross rate of intertemporal discounting of utility

- 3 Important concept: marginal rate of substitution between  $c_t$  and  $c_{t+1}$

$$MRS_{t,t+1} = -\frac{\partial U / \partial c_t}{\partial U / \partial c_{t+1}} = -\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})}$$

# The typical household problem: preferences



## The typical household problem: constraints

- ① The constraints for the two periods of time are

$$\begin{aligned}c_t + a_{t+1} &\leq w_t \\c_{t+1} &\leq (1 + r_{t+1})a_{t+1} + w_{t+1}\end{aligned}$$

$a_{t+1}$  – savings at  $t$  and transformed into a financial asset at  $t + 1$   
 $w_t$  – wages or income at  $t$ ,  $r_{t+1}$  – interest rate (return on financial assets)

- ② The consolidated intertemporal constraint: eliminate  $a_{t+1}$ . **Expressed in current values**

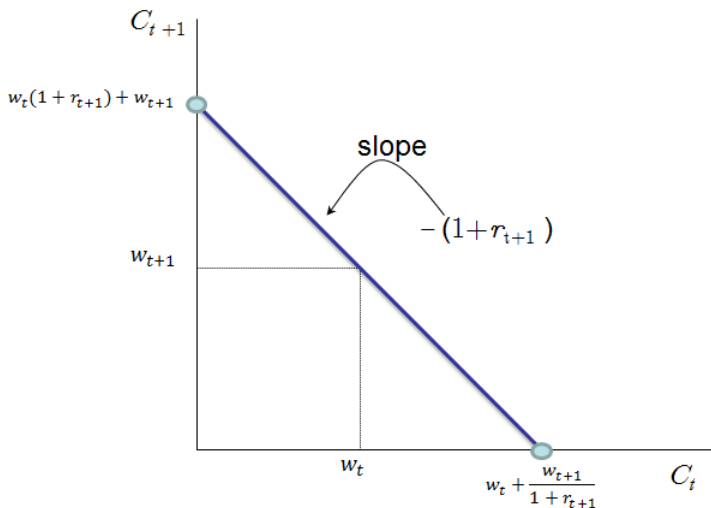
$$\underbrace{c_t + \frac{c_{t+1}}{1 + r_{t+1}}}_{\text{value of intertemporal consumption}} \leq \underbrace{w_t + \frac{w_{t+1}}{1 + r_{t+1}}}_{\text{value of intertemporal income}}$$

- ③ The consolidated intertemporal constraint: eliminate  $a_{t+1}$ . **Expressed in future values**

$$(1 + r_{t+1})c_t + c_{t+1} \leq (1 + r_{t+1})w_t + w_{t+1}$$



# The typical household problem: constraints



## The maximization of utility

- 1 It implies the finding of the optimal levels of consumption  $(c_t^*, c_{t+1}^*)$  and of savings  $(a_{t+1}^*)$  over time
- 2 This can be done in two ways: graphically, algebraically
- 3 The slope of the intertemporal budget constraint is given by

$$-(1 + r_{t+1})$$

- 4 The Marginal Rate of Substitution is given by

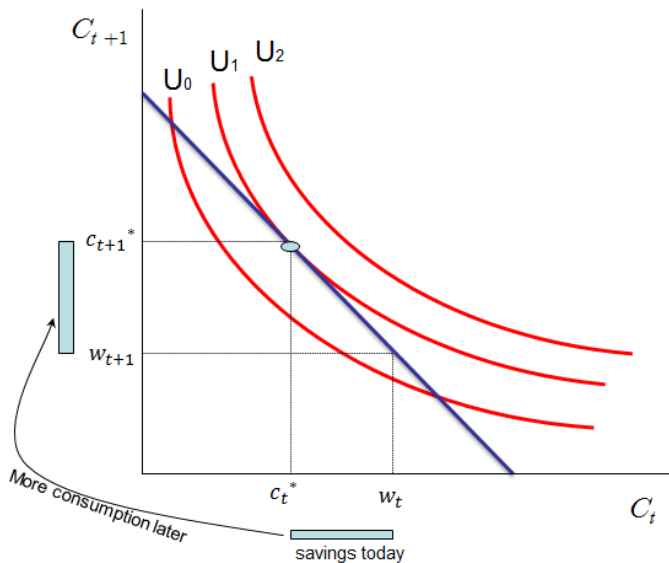
$$MRS_{t,t+1} = -\frac{\partial U / \partial c_t}{\partial U / \partial c_{t+1}} = -\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})}$$

- 5 Equalizing both, we get

$$-\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})} = -(1 + r_{t+1})$$

$$u'(c_t) = (1 + r_{t+1}) \cdot \beta \cdot u'(c_{t+1})$$

# The typical household problem: constraints



## II – The Lagrange Method

# What is it?

- 1 In everything similar to the Lagrangean function that you learned in every intermediate course in Microeconomics
- 2 Very useful if time is discrete
- 3 Very intuitive
- 4 Highly used in modern macroeconomics everywhere
- 5 We have:
  - 1 An objective function
  - 2 A constraint (or various constraints)
  - 3 Viability conditions (to assure the optimal values)

## Formally speaking

- 1 Objective: to maximize utility (now, intertemporal)

$$\max_{c_t, c_{t+1}, a_{t+1}} u(c_t) + \beta \cdot u(c_{t+1})$$

- 2 There are two constraints:

$$c_t + a_{t+1} \leq w_t$$

$$c_{t+1} \leq (1 + r_{t+1})a_{t+1} + w_{t+1}$$

- 3 Viability conditions are:  $c_t, c_{t+1} \geq 0$

- 4 Formal setting of the problem

$$\max_{c_t, c_{t+1}, a_{t+1}} u(c_t) + \beta \cdot u(c_{t+1})$$

$$\text{s.t. } c_t + a_{t+1} \leq w_t$$

$$c_{t+1} \leq (1 + r_{t+1})a_{t+1} + w_{t+1}$$

$$c_t, c_{t+1} \geq 0$$

# The Lagrangean

- 1 The Lagrangean function is written as

$$\mathcal{L} = u(c_t) + \beta \cdot u(c_{t+1}) + \lambda_t(w_t - c_t - a_{t+1}) + \lambda_{t+1}[w_{t+1} + (1 + r_{t+1})a_{t+1} - c_{t+1}]$$

$$c_t, c_{t+1} \geq 0,$$

$$\lambda_t, \lambda_{t+1} \geq 0$$

- 2 How many unknowns?
- 3 Five unknowns:  $c_t, c_{t+1}, a_{t+1}, \lambda_t, \lambda_{t+1}$
- 4 Five First Order Conditions (**FOC**)

# First Order Conditions

- ① The 5 FOCs are

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow u'(c_t) - \lambda_t = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = 0 \Rightarrow \beta \cdot u'(c_{t+1}) - \lambda_{t+1} = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = 0 \Rightarrow -\lambda_t + \lambda_{t+1}(1 + r_{t+1}) = 0 \quad (3)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \Rightarrow w_t - c_t - a_{t+1} = 0 \quad (4)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_{t+1}} = 0 \Rightarrow w_{t+1} + (1 + r_{t+1})a_{t+1} - c_{t+1} = 0 \quad (5)$$

- ② Now there is always a very useful trick: **you can eliminate the  $\lambda$ 's.**
- ③ Then simplify
- ④ Let's do it.



## Simplification

- 1 From the first two FOCs (FOC1, FOC2)

$$\begin{aligned}\partial \mathcal{L} / \partial c_t &= 0 \Rightarrow u'(c_t) - \lambda_t = 0 \\ \partial \mathcal{L} / \partial c_{t+1} &= 0 \Rightarrow \beta \cdot u'(c_{t+1}) - \lambda_{t+1} = 0\end{aligned}$$

- 2 we get

$$\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})} = \frac{\lambda_t}{\lambda_{t+1}} \quad (6)$$

- 3 From FOC3  $-\lambda_t + \lambda_{t+1}(1 + r_{t+1}) = 0$  we get

$$\lambda_t / \lambda_{t+1} = 1 + r_{t+1} \quad (7)$$

- 4 Equalizing the two previous eq., we get our already known result

$$\underbrace{\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})}}_{MRS_{t,t+1}} = \underbrace{(1 + r_{t+1})}_{\text{relative price}}$$

- 5 So, either graphically, or by the Lagrange method we arrived at the

# The Euler Equation

- 1 We have just got this result

$$\underbrace{\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})}}_{MRS_{t,t+1}} = \underbrace{(1 + r_{t+1})}_{\text{relative price}}$$

- 2 In an equivalent way, known as the **EULER EQUATION**

$$u'(c_t) = (1 + r_{t+1}) \cdot \beta \cdot u'(c_{t+1})$$

- 3 The Euler equation is extremely important in modern macroeconomics:

- 1 allows to incorporate microfoundations into macroeconomics (preferences of the demand side)
- 2 allows to measure the welfare of different macroeconomic policies

- 4 Three possible cases can be easily spotted.

$$(1 + r_{t+1}) \cdot \beta = 1 \Rightarrow u'(c_t) = u'(c_{t+1})$$

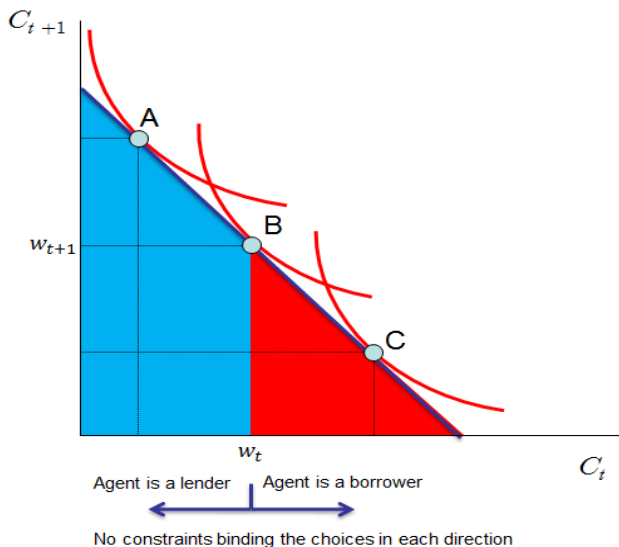
$$(1 + r_{t+1}) \cdot \beta \leq 1 \Rightarrow u'(c_t) \leq u'(c_{t+1})$$

# III – Liquidity constraints

# The liquidity constraint problem

- 1 Up to now, we have assumed that the agents face **no debt constraint**
- 2 **Any transaction** between current and future consumption is possible.
- 3 Whenever one agent wants to increase current consumption, there is always some other agent that wants the opposite.
- 4 If there is a mismatch between the savings of some and the debt intentions of others, the relative price of intertemporal consumption will change  $(1 + r_{t+1})$ , so that equilibrium will be reestablished again
- 5 That is, up to now: **MARKETS WERE COMPLETE**
- 6 What happens if markets are not complete? If some agents are faced with debt constraints?

# The no-liquidity constraint case



## Binding versus Non-Binding constraints

- ① Assume that if a household has negative savings

$$a_{t+1}^-$$

- ② ... **can not borrow more than** a certain value at  $t$

$$a_{t+1}^- \leq A$$

- ③ Solution: totally equal to the solution with complete markets

- ④ The only difference now is that these constraints should apply

$$\begin{aligned} a_{t+1}^- &\leq A \\ c_t, c_{t+1} &\geq 0 \end{aligned}$$

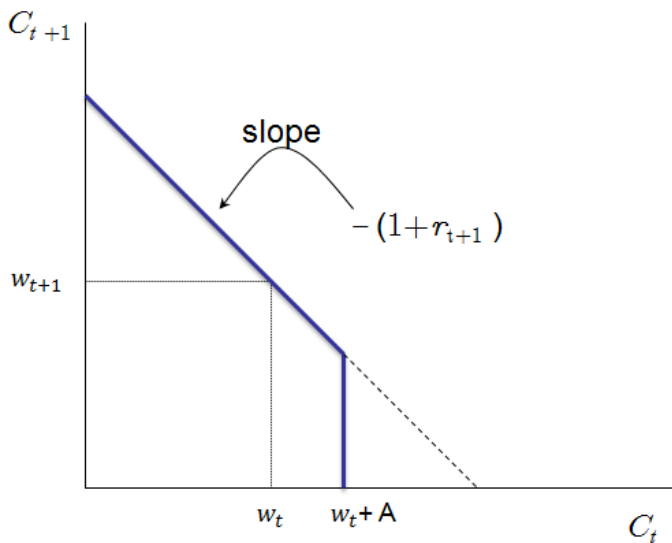
- ⑤ No Ponzii game: at  $t+1$  the borrower must be able to repay debt

$$(1 + r_{t+1}) A < w_{t+1}$$

- ⑥ Binding versus Non-Binding constraints:

$$a_{t+1}^- = A \quad , \quad \text{Binding}$$

$$a_{t+1}^- < A \quad , \quad \text{Non-Binding}$$



## A binding constraint

- 1 This occurs if

$$a_{t+1}^- = A$$

- 2 The agent wants to consume more at  $t$ , but he cannot do it because he is financially constrained

$$MRS > \text{relative price}$$

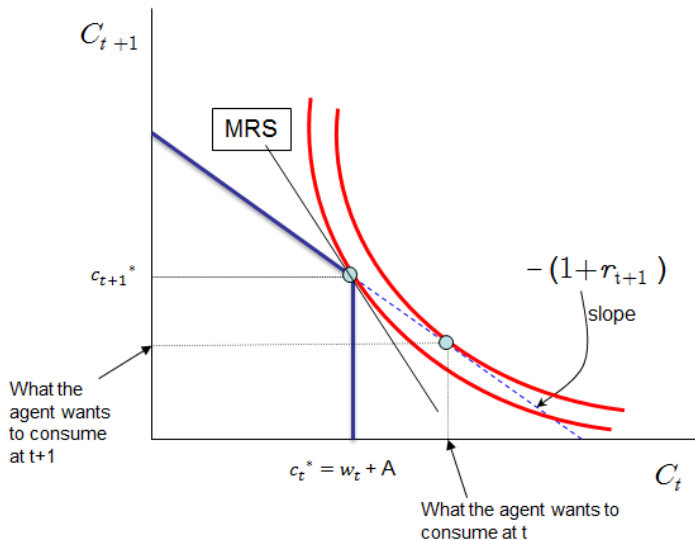
- 3 Mathematically:

$$\underbrace{\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})}}_{MRS_{t,t+1}} > \underbrace{(1 + r_{t+1})}_{\text{relative price}}$$

- 4 **Implications? Economic inefficiency:** the welfare of our constrained agent is lower than without the binding constraint.



# An example of a binding constrain



## A non-binding constraint

- 1 This occurs if

$$a_{t+1}^- < A$$

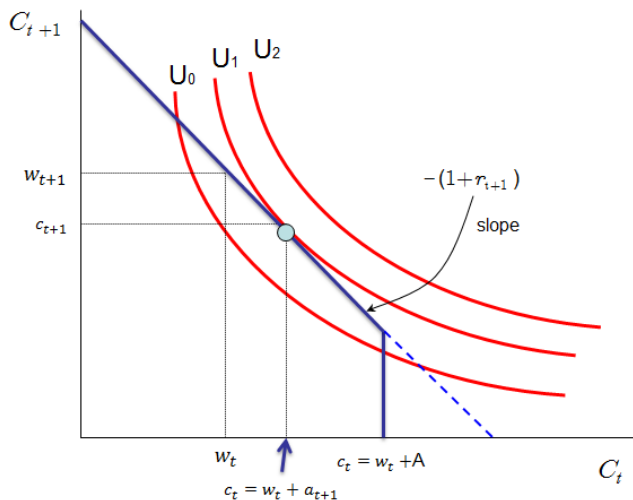
- 2 Solution totally equal to the solution without debt constraints
- 3 No economic inefficiency

$$MRS = \text{relative price}$$

- 4 Mathematically:

$$\underbrace{\frac{u'(c_t)}{\beta \cdot u'(c_{t+1})}}_{MRS_{t,t+1}} = \underbrace{(1 + r_{t+1})}_{\text{relative price}}$$

# A non-binding constraint example



# IV – Endogenous labor supply

## The two fundamental assumptions: up to now

- 1 The results obtained depended on two fundamental assumptions.
- 2 **Assumption 1.** The level of income in both periods was taken as exogenous

$w_t, w_{t+1}$  were given

- 3 **Assumption 2.** The level of leisure (and the labor effort) was taken as exogenous

$\ell = \bar{\ell}$ , labor effort as given

$1 - \ell$ , leisure time given

- 4 However, in reality we know that the two ingredients in the previous assumptions are linked

wages can increase, if the labor effort increases

- 5 So what happens to intertemporal utility?

# The new intertemporal problem

- ① How can this link be integrated into the model? Through the utility function.
- ② Working more hours
  - ① leads to higher income, which leads to higher consumption, which implies higher utility levels
  - ② However, it also implies less leisure time, which implies lower utility
  - ③ There is a trade-off here
- ③ So now we will have two intertemporal decisions, not just one:
  - ① Not only, should I consume more now, or more in the future?
  - ② But also, should I work more hours, or should I enjoy more leisure?

$\uparrow \ell \Rightarrow \uparrow \text{income} \Rightarrow \uparrow \text{consumption} \Rightarrow \uparrow \text{utility}$

$\uparrow \ell \Rightarrow \downarrow \text{leisure} \Rightarrow \downarrow \text{utility}$

## Setting the new intertemporal problem

### 1 Symbols:

$\ell$  : labor effort;  $1 - \ell$  : leisure

$\beta$  : intertemporal discount factor

### 2 Agents preferences

$$U(c, \ell) = u(c_t, 1 - \ell_t) + \beta \cdot u(c_{t+1}, 1 - \ell_{t+1})$$

*or in another way*

$$U(c, \ell) = u(c_t, \ell_t) + \beta \cdot u(c_{t+1}, \ell_{t+1})$$

### 3 Let's impose some useful conditions on the utility function

$$\begin{pmatrix} u'_c > 0 & , & u''_c < 0 \\ u'_{1-\ell} > 0 & , & u''_{1-\ell} < 0 \\ u'_\ell < 0 & , & u''_\ell > 0 \end{pmatrix}$$

## Setting the new intertemporal problem (cont.)

### 1 Intertemporal constraints

$$c_t + a_{t+1} \leq w_t \ell_t$$

$$c_{t+1} \leq (1 + r_{t+1})a_{t+1} + w_{t+1} \ell_{t+1}$$

### 2 Consolidated constraint: eliminate $a_{t+1}$ . Expressed in current values

$$\underbrace{c_t + \frac{c_{t+1}}{1 + r_{t+1}}}_{\text{value of intertemporal consumption}} \leq \underbrace{w_t \ell_t + \frac{w_{t+1} \cdot \ell_{t+1}}{1 + r_{t+1}}}_{\text{value of intertemporal income}}$$

### 3 Maximizing utility

$$\max_{c_t, c_{t+1}, \ell_t, \ell_{t+1}, a_{t+1}} u(c_t, 1 - \ell_t) + \beta \cdot u(c_{t+1}, 1 - \ell_{t+1})$$

$$\text{s.t. } c_t + a_{t+1} \leq w_t \ell_t$$

$$c_{t+1} \leq (1 + r_{t+1})a_{t+1} + w_{t+1} \ell_{t+1}$$

$$c_t, c_{t+1} \geq 0, \ell_t, \ell_{t+1} \in (0, 1)$$



# The Lagrangean function

- 1 The Lagrangean function is written as

$$\mathcal{L} = u(c_t, 1 - \ell_t) + \beta \cdot u(c_{t+1}, 1 - \ell_{t+1}) + \lambda_t (w_t \ell_t - c_t - a_{t+1}) - \lambda_{t+1} [w_{t+1} \ell_{t+1} + (1 + r_{t+1})a_{t+1} - c_{t+1}]$$

$$c_t, c_{t+1} \geq 0, \quad \ell_t, \ell_{t+1} \in (0, 1), \quad \lambda_t, \lambda_{t+1} \geq 0$$

- 2  $\lambda_t$  are the Lagrange multipliers  
 3 How many unknowns? Seven.

$$c_t, c_{t+1}, \ell_t, \ell_{t+1}, a_{t+1}, \lambda_t, \lambda_{t+1}$$

- 4 How many First Order Conditions? Seven

# The First Order Conditions (FOCs)

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow u'_{c_t} - \lambda_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = 0 \Rightarrow \beta \cdot u'_{c_{t+1}} - \lambda_{t+1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \ell_t} = 0 \Rightarrow -u'_{1-\ell_t} - \lambda_t w_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial \ell_{t+1}} = 0 \Rightarrow -\beta \cdot u'_{1-\ell_{t+1}} + \lambda_{t+1} w_{t+1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = 0 \Rightarrow -\lambda_t + \lambda_{t+1}(1 + r_{t+1}) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \Rightarrow w_t \ell_t - c_t - a_{t+1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_{t+1}} = 0 \Rightarrow w_{t+1} \ell_{t+1} + (1 + r_{t+1}) a_{t+1} - c_{t+1} = 0$$

# The Euler Equation

- 1 From the first two FOCs

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow u'_{c_t} - \lambda_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = 0 \Rightarrow \beta \cdot u'_{c_{t+1}} - \lambda_{t+1} = 0$$

- 2 we get

$$\frac{u'_{c_t}}{\beta \cdot u'_{c_{t+1}}} = \frac{\lambda_t}{\lambda_{t+1}}$$

- 3 From FOC5  $-\lambda_t + \lambda_{t+1}(1 + r_{t+1}) = 0$  we get

$$\frac{\lambda_t}{\lambda_{t+1}} = 1 + r_{t+1}$$

- 4 Equalizing the two previous eq., we get our already known **Euler Equation**

$$u'_{c_t} = (1 + r_{t+1}) \cdot \beta \cdot u'_{c_{t+1}} \quad (8)$$

## Optimal intertemporal labor effort

- 1 Use FOC3, FOC4, and combine, respectively, with FOC1 and FOC2
- 2 We will get

$$\frac{u'_{1-\ell_t}}{u'_{c_t}} = w_t$$

$$\frac{u'_{1-\ell_{t+1}}}{u'_{c_{t+1}}} = w_{t+1} 1$$

- 3 Simplifying, we get a static condition for each period that optimizes the labor effort

$$\underbrace{u'_{c_t}}_{\text{marginal utility of } c} = \underbrace{\frac{u'_{1-\ell_t}}{w_t}}_{\text{marginal utility of leisure weighted by its price}}$$

$$\underbrace{u'_{c_{t+1}}}_{\text{marginal utility of } c} = \underbrace{\frac{u'_{1-\ell_{t+1}}}{w_{t+1}}}_{\text{marginal utility of leisure weighted by its price}}$$

## Summary: the tree intertemporal optimal conditions

- 1 One dynamic condition: the Euler equation, should I consume more now, or should I consume more in the future?

$$u'_{c_t} = (1 + r_{t+1}) \cdot \beta \cdot u'_{c_{t+1}}$$

- 2 Two static conditions: should I work more, or should I work less?

$$u'_{c_t} = \frac{u'_{1-\ell_t}}{w_t}$$

$$u'_{c_{t+1}} = \frac{u'_{1-\ell_{t+1}}}{w_{t+1}}$$

## Example 1

- 1 Assume that agents only work in the first period
- 2 Assume that the utility function is logarithmic

$$U(c, \ell) = \ln c_t + \ln(1 - \ell_t) + \beta \ln c_{t+1}$$

- 3 Utility maximization

$$\max_{c_t, c_{t+1}, \ell_t, a_{t+1}} \ln c_t + \ln(1 - \ell_t) + \beta \ln c_{t+1}$$

$$\text{s.t.} \quad c_t + a_{t+1} \leq w_t \ell_t$$

$$c_{t+1} \leq (1 + r_{t+1})a_{t+1} + w_{t+1} \cdot \underbrace{\ell_{t+1}}_0$$

$$c_t, c_{t+1} \geq 0, \quad \ell_t \in (0, 1)$$

## Example 1 (cont.)

- ① Applying directly the Euler equation

$$u'_{c_t} = (1 + r_{t+1}) \cdot \beta \cdot u'_{c_{t+1}}$$

- ② We get

$$\frac{1}{c_t} = (1 + r_{t+1}) \cdot \beta \cdot \frac{1}{c_{t+1}} \quad (9)$$

- ③ That is

$$c_{t+1} = \beta (1 + r_{t+1}) c_t$$

- ④ Now applying the first static condition  $u'_{c_t} \cdot w_t = u'_{1-\ell_t}$ , we get

$$\frac{1}{c_t} \cdot w_t = \frac{1}{1 - \ell_t}$$

$$c_t = w_t (1 - \ell_t) \quad (10)$$

## Example 1 (cont.)

- ① Using the intertemporal consolidated constraint (notice that  $l_{t+1} = 0$ )

$$c_t + \frac{c_{t+1}}{1 + r_{t+1}} \leq w_t l_t$$

- ② ...together with the Euler Equation (9), we get

$$c_t = \frac{1}{1 + \beta} w_t l_t \quad (11)$$

- ③ Now we can determine the optimal intertemporal value of labor effort
- ④ Just combine eq(10) and eq(11) and we get

$$l_t^* = \frac{1 + \beta}{2 + \beta}$$



## Example 1: final results

- ① We have just got the optimal level of labor effort for period  $t$

$$\ell_t^* = \frac{1 + \beta}{2 + \beta}$$

- ② Applying back into eq(11), we get

$$c_t^* = \frac{1}{2 + \beta} w_t$$

- ③ And once we know  $c_t^*$ , we can easily determine  $c_{t+1}^*$

$$c_{t+1}^* = \left( \frac{w_t}{2 + \beta} \right) \beta (1 + r_{t+1})$$

- ④ **Important conclusion: the supply of labor is inelastic with respect to wages.** ( $\ell_t^*$  is independent from  $w_t$ )

## Example 2

- 1 Assume now that agents have an initial level of wealth  $a_t > 0$ .
- 2 Moreover, they continue working only in period  $t$
- 3 We will see that the conclusion in the previous slide is totally reversed.
- 4 The problem now looks like

$$\max_{c_t, c_{t+1}, \ell_t, a_{t+1}} \ln c_t + \ln(1 - \ell_t) + \beta \ln c_{t+1}$$

$$\text{s.t. } c_t + a_{t+1} \leq w_t \ell_t + a_t(1 + r_t)$$

$$c_{t+1} \leq (1 + r_{t+1})a_{t+1} + w_{t+1} \cdot \underbrace{\ell_{t+1}}_0$$

$$c_t, c_{t+1} \geq 0, \quad \ell_t \in (0, 1)$$

## Example 2 (cont.)

- ① From the two FOCs with respect to  $\ell_t, \ell_{t+1}$ , we will get

$$w_t (1 - \ell_t) (1 + \beta) = w_t \ell_t + a_t (1 + r_t)$$

- ② From where we can obtain the optimal level of labor effort

$$\ell_t^* = \frac{1}{2 + \beta} \left[ 1 + \beta - \frac{a_t (1 + r_t)}{w_t} \right]$$

- ③ Therefore, in this case we have

$$\uparrow w_t \Rightarrow \uparrow \ell_t^* \quad , \quad \uparrow r_t \Rightarrow \downarrow \ell_t^* \quad , \quad \uparrow a_t \Rightarrow \downarrow \ell_t^*$$

# V – Bibliography

# Bibliography

If you are able to read Spanish, the next reading is very,very good:



J.C. Conesa and C. Garriga (2011). *Teoria Económica del Capital y la Renta*, Universitat Autònoma de Barcelona. *Read chapter 4 (for points 1, 2 and 3) and Chapter 5 (for point 5). Follow the slides, as some parts of each chapter are skipped without loss of any relevant information for the course.*

If you prefer a text in English, the next one is concise but very good:



Martin Boileau (2001). "Two Period Economies: A Review", University of Colorado, Boulder. *For those of are not fluent in reading in Spanish, this is a good alternative to the reference above.*

## Bibliography (cont.)

Another short text in English, concise, good, and with too many questions along the text (do not worry about these questions) is:

 S. Shi (2009). "Two Period Economies", University of Toronto, Toronto.

If you have not dealt with any kind of this stuff before in your studies and you feel a little bit uneasy, *for an introductory treatment of the issues discussed in this week. See Chapters 4 and 8, at the expenses of having to read a much larger number of pages.*

 Stephen Williamson (2011). *Macroeconomics*, Pearson, New York.