The New Keynesian Model: Simulations

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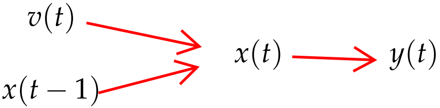
Summary

1. Linearization and the Blanchard-Kahn approach
2. Simple examples of the NKM in the Blanchard-Kahn framework
3. The most used version of the NKM: demand and supply shocks
4. Recommende reading

I – Linearization and the Blanchard-Kahn approach

The basic idea behind the B-K method

1. Apply the Jordan decomposition to transform our complicated models into two distinct blocks:
   1. The block including only predetermined variables
   2. The other block including only forward looking variables
2. Then we can apply exactly the same strategy followed in part I.
   1. Iterate forward
   2. Firstly, the predetermined block
   3. Then, the forward looking block.
3. And we get the same type of results



Blanchard, O., and C.M. Kahn. (1980). The solution of linear difference models under rational expectations. *Econometrica* 48(5), 1305–1311.

The Jordan decomposition

1. Compute the Jordan canonical form (also called Jordan normal form) of a symbolic or numeric matrix A
2. Our model comprises: a set of predetermined variables (x_t), a set of forward looking variables (y_t), and a set of exogenous shocks (v_t)
3. Write the model in state space form

$$\left[\begin{array}{c} w_{t+1} \\ E_t y_{t+1} \end{array}\right] = A \left[\begin{array}{c} w_t \\ y_t \end{array}\right] + B v_{t+1} \tag{1}$$

4. The Jordan decomposition of (A)

$$A = P \Lambda P^{-1}$$

5. Λ is a diagonal matrix with the **eigenvalues** of A along its leading diagonal and zeros in the remaining entries.
6. P contains the inverse matrix of the generalized **eigenvectors** of A as columns

The model with the Jordan decomposition

1. Apply the decomposition

$$\begin{bmatrix} w_{t+1} \\ E_t y_{t+1} \end{bmatrix} = P\Lambda P^{-1} \begin{bmatrix} w_t \\ y_t \end{bmatrix} + Bv_{t+1} \tag{2}$$

2. Multiply both sides by P^{-1}

$$P^{-1} \begin{bmatrix} w_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \Lambda P^{-1} \begin{bmatrix} w_t \\ y_t \end{bmatrix} + \underbrace{P^{-1}B}_{=R} \cdot v_{t+1} \tag{3}$$

The model with the Jordan decomposition

1. Partition P^{-1} and Λ to get

$$\underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w_{t+1} \\ E_t y_{t+1} \end{bmatrix}}_{Et \begin{bmatrix} \tilde{w}_{t+1} \\ \tilde{y}_{t+1} \end{bmatrix}} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w_t \\ y_t \end{bmatrix}}_{\begin{bmatrix} \tilde{w}_t \\ \tilde{y}_t \end{bmatrix}} + \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} v_{t+1} \tag{4}$$

2. So our transformed model looks much easier now

$$\begin{bmatrix} \tilde{w}_{t+1} \\ E_t \tilde{y}_{t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \tilde{w}_t \\ \tilde{y}_t \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} v_{t+1}$$

The two decoupled blocks

Transformed model written down as a set of decoupled equations

$$\tilde{w}_{t+1} = \Lambda_1 \tilde{w}_t + R_1 v_{t+1} \qquad \text{(Stable block)}$$

$$E_t \tilde{y}_{t+1} = \Lambda_2 \tilde{y}_t + R_2 v_{t+1} \qquad \text{(Unstable block)}$$

1. We can now apply our old strategy
   1. Solve the unstable transformed block forward and get: $\tilde{y}_t^*$
   2. Solve the stable transformed block backwards and get: $\tilde{w}_t^*$
2. Insert the results back into the original problem

Solving the unstable block

1. Iterating forward this block, we get

$$E_t \tilde{y}_{t+n} = (\Lambda_2)^n \tilde{y}_t$$

2. If we have

$$|\Lambda_2| > 1$$

3. Then, the only stable solution will be

$$\tilde{y}_t^* = 0, \ \forall t$$

4. Now from our definition in eq. (4), we know that

$$\tilde{y}_t^* = P_{21} \cdot w_t^* + P_{22} \cdot y_t^* = 0$$

5. From which

$$y_t^* = \left[-P_{22}^{-1} P_{21} \right] \cdot w_t^* \tag{5}$$

6. **Notice that this our old result:** forward looking variables depending upon predetermined ones.

Solving the stable block

1. Iterating forward this block, we get

$$\tilde{w}_{t+n} = (\Lambda_1)^n \tilde{w}_t \quad , \qquad |\Lambda_1| < 1$$

2. If we assume that

$$|\Lambda_1| < 1$$

3. The process is stable, and from eq. (4), we get

$$\tilde{w}_t^* = P_{11} \cdot w_t^* + P_{12} \cdot y_t^* \tag{6}$$

4. Now insert eq. (5) into (6), and get

$$\tilde{w}_t^* = \underbrace{\left[P_{11} - P_{12} P_{22}^{-1} P_{21} \right]}_{D} \cdot w_t^* \tag{7}$$

Solving the stable block (cont.)

1. But as from eq. (Stable block), we have

$$\tilde{w}_{t+1} = \Lambda_1 \tilde{w}_t + R_1 v_{t+1}$$

2. And as from eq.(7) we have

$$\tilde{w}_t^* = D \cdot w_t^*$$

3. Then

$$D \cdot w_{t+1}^* = \tilde{w}_{t+1}^* \qquad \tilde{w}_t^* = D \cdot w_t^*$$

$$\tilde{w}_{t+1} = \Lambda_1 \tilde{w}_t + R_1 v_{t+1}$$

4. From which we finally get

$$w_{t+1}^* = \left[D^{-1} \Lambda_1 D \right] w_t^* + \left[D^{-1} R_1 \right] v_{t+1} \tag{8}$$

Summarizing

1. Write down your model in state space form
2. Apply the Jordan decomposition
3. Decouple the system into two blocks
4. Make sure the eigenvalues guarantee a unique and stable solution to the model.
5. End up with the two fundamental results

$$
\begin{aligned}
y_t^* &= \left[-P_{22}^{-1} P_{21}\right] \cdot w_t^* \\
w_{t+1}^* &= \left[D^{-1} \Lambda_1 D\right] w_t^* + \left[D^{-1} R_1\right] v_{t+1}
\end{aligned}
$$

with $D = P_{11} - P_{12} \left(P_{22}\right)^{-1} P_{21}$

II – Simple examples of the NKM in the Blanchard-Kahn framework

The NKM with a very naive MP rule and two shocks

1. The baseline version includes five equations:

$$\begin{aligned} x_t &= E_t x_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1}\right) && \text{(IS)} \\ \pi_t &= \beta \cdot E_t \pi_{t+1} + \kappa x_t + u_t && \text{(AS)} \\ i_t &= \delta \pi_t + v_t && \text{(Monetary Policy Rule)} \\ v_{t+1} &= \rho_v v_t + \epsilon^v_{t+1} && \text{(Shock v)} \\ u_{t+1} &= \rho_u u_t + \epsilon^u_{t+1} && \text{(Shock u)} \end{aligned}$$

   with $\kappa = (1-\omega)(1-\beta\omega)/(\alpha\omega)$

2. Can we apply the same strategy?

$$\begin{aligned} \text{Predetermined} &: \quad v_t, u_t \\ \text{Forward looking} &: \quad x_t, \pi_t \end{aligned}$$

3. A unique and stable equilibrium requires: $|\lambda_1, \lambda_2| < 1; |\lambda_3, \lambda_4| > 1$

The NKM in state space

1. Take the following calibration

$$\beta = 0.99, \sigma = 1, \delta = 1.5, \rho_v = 0.8, \rho_u = 0.5, \alpha = 3, \omega = 0.5$$

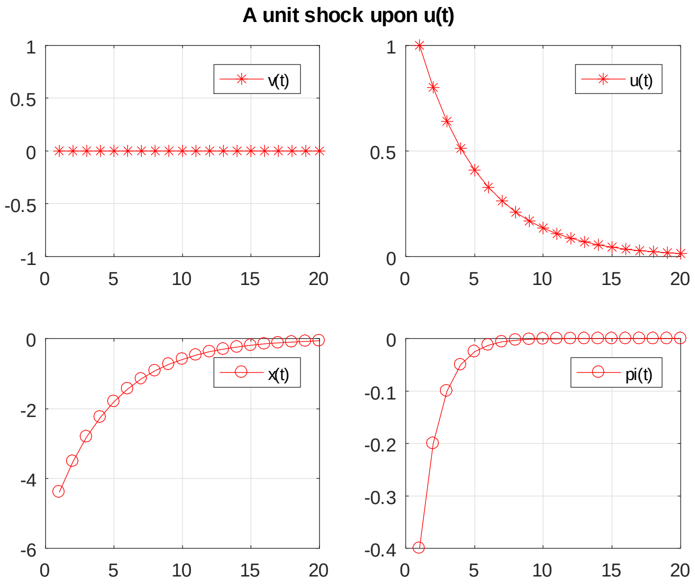
2. The model can be written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{\sigma} \\ 0 & 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} v_{t+1} \\ u_{t+1} \\ E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} =$$

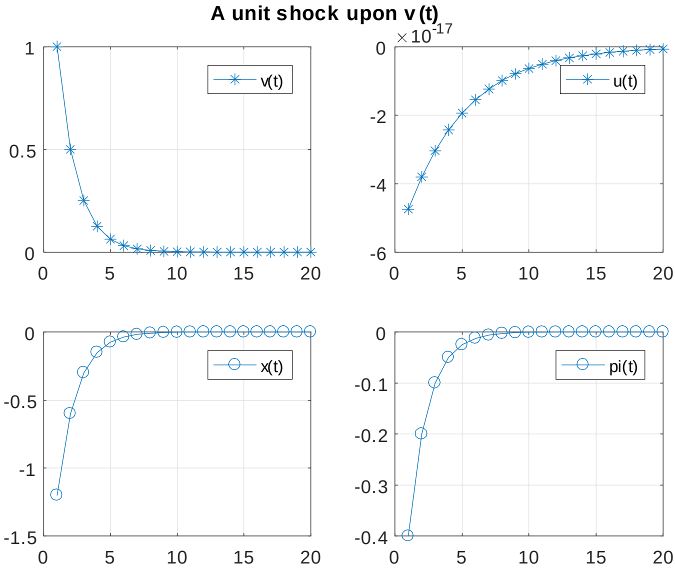
$$\begin{bmatrix} \rho_v & 0 & 0 & 0 \\ 0 & \rho_u & 0 & 0 \\ \frac{1}{\sigma} & 0 & 1 & \frac{1}{\sigma}\delta \\ 0 & -1 & -\kappa & 1 \end{bmatrix} \begin{bmatrix} v_t \\ u_t \\ x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon^v_{t+1} \\ \epsilon^u_{t+1} \end{bmatrix}$$

3. Look at the routine **NKM_Topics_Macro_V2.m** and the IRF that come out of the model

The NKM: IRF from a shock upon u(t)



The NKM: IRF from a shock upon v(t)



The NKM with shocks to the natural real interest rate

1. The baseline version includes five equations:

$$
\begin{aligned}
x_t &= E_t x_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1} - r_t^n\right) && \text{(IS)} \\
\pi_t &= \beta \cdot E_t \pi_{t+1} + \kappa x_t && \text{(AS)} \\
i_t &= \phi_\pi \pi_t + \phi_x x_t && \text{(Monetary policy rule)} \\
r_t^n &= \rho r_{t-1}^n + \epsilon_t && \text{(Natural real interest rate)}
\end{aligned}
$$

with

$$
\kappa = \frac{(1-\theta)(1-\theta\beta)}{\theta(\sigma+\psi)}
$$

2. Notice that in this case

$$
\begin{aligned}
\text{Predetermined} &: \quad r_t^n \\
\text{Forward looking} &: \quad x_t, \pi_t
\end{aligned}
$$

The NKM with shocks to the natural real interest rate

1. The model can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{\sigma} \\ 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} r^n_{t+1} \\ E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} =$$

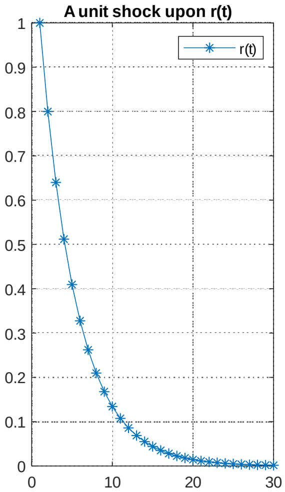
$$\begin{bmatrix} \rho & 0 & 0 \\ -\frac{1}{\sigma} & \frac{\sigma+\phi_x}{\sigma} & \frac{\phi_\pi}{\sigma} \\ & -\kappa & 1 \end{bmatrix} \begin{bmatrix} r^n_t \\ x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [\epsilon_{t+1}]$$

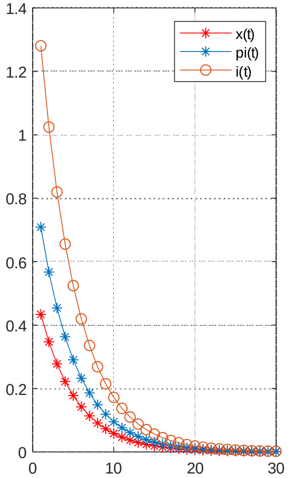
2. With the following calibration

$$\beta = 0.99, \sigma = 1, \phi_\pi = 1.5, \phi_x = 0.5, \rho = 0.8, \theta = 2/3, \psi = 1$$

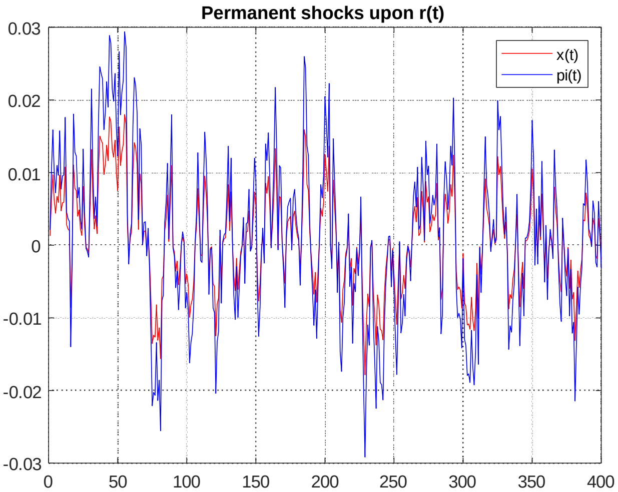
3. See the routine: **Exercise_NKM_Standard_Approach_IRF_TimeSeries_More.m**
4. You will get IRF, time series and more

The NKM with shocks to the natural real interest rate

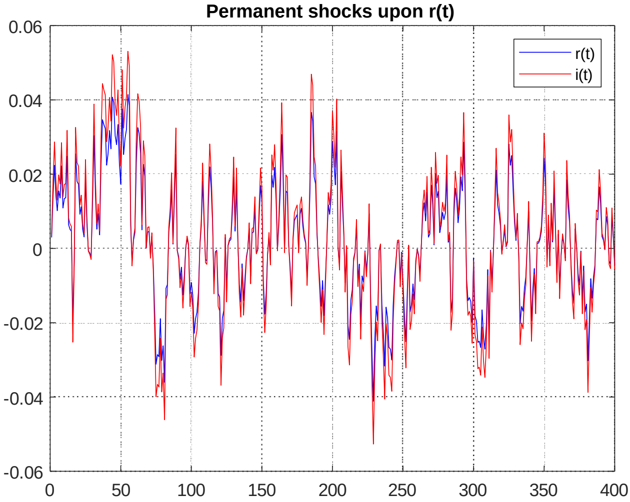




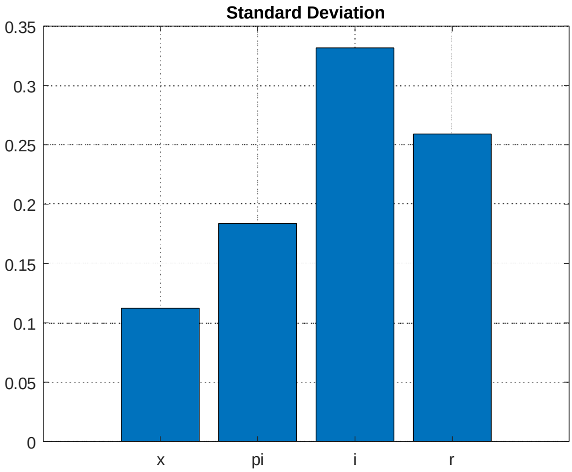
The NKM with shocks to the natural real interest rate



The NKM with shocks to the natural real interest rate



The NKM with shocks to the natural real interest rate



III – The most used version of the NKM: demand and supply shocks

The NKM with demand ans supply shocks

1. The baseline version includes five equations:

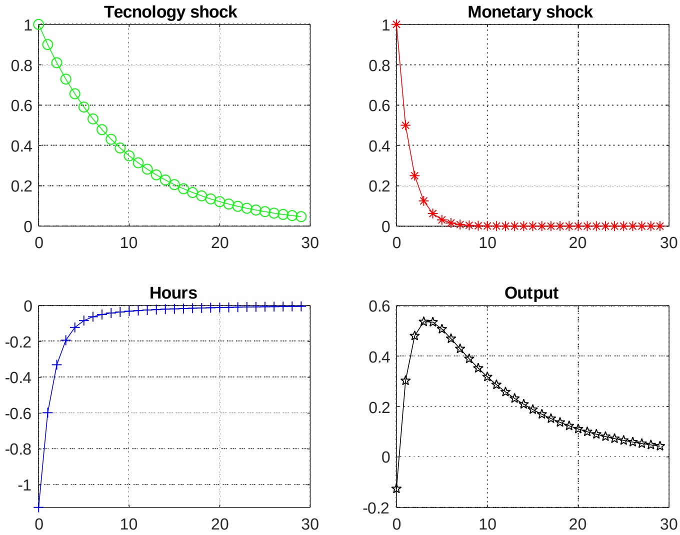
$$\begin{aligned} x_t &= E_t x_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1} - r_t^n\right) && \text{(IS)} \\ \pi_t &= \beta \cdot E_t \pi_{t+1} + \kappa x_t && \text{(AS)} \\ i_t &= \phi_\pi \pi_t + \phi_x x_t + v_t && \text{(Monetary policy rule)} \\ r_t^n &= \rho \psi E_t a_{t+1} && \text{(Natural real interest rate)} \\ v_{t+1} &= \rho_v v_t + \epsilon_{t+1}^v && \text{(Demand shock)} \\ a_{t+1} &= \rho_a a_t + \epsilon_{t+1}^a && \text{(Supply shock)} \end{aligned}$$

   with

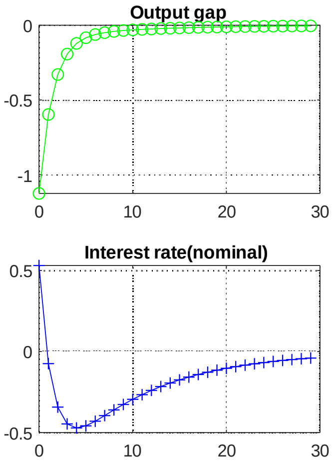
$$\kappa = \frac{(1-\theta)(1-\theta\beta)}{\theta(\sigma+\varphi)}$$

2. Simulate the model using the routine **Baseline_NK_model.m** (written by Z. Enders) and compare the output from the two different shocks..

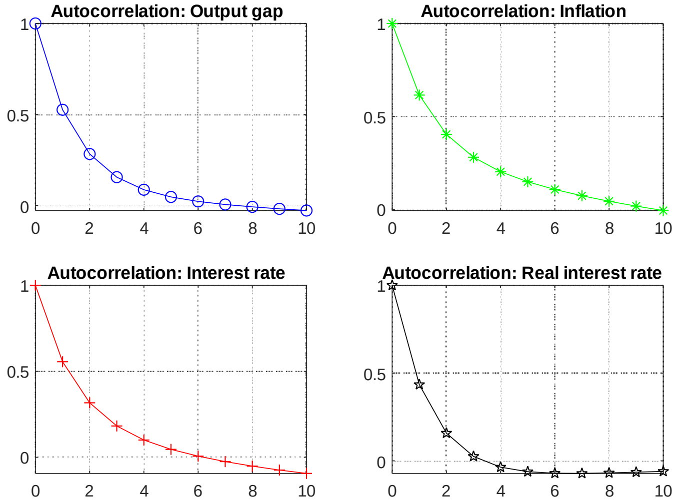
The NKM: IRF from both demand and shock shocks



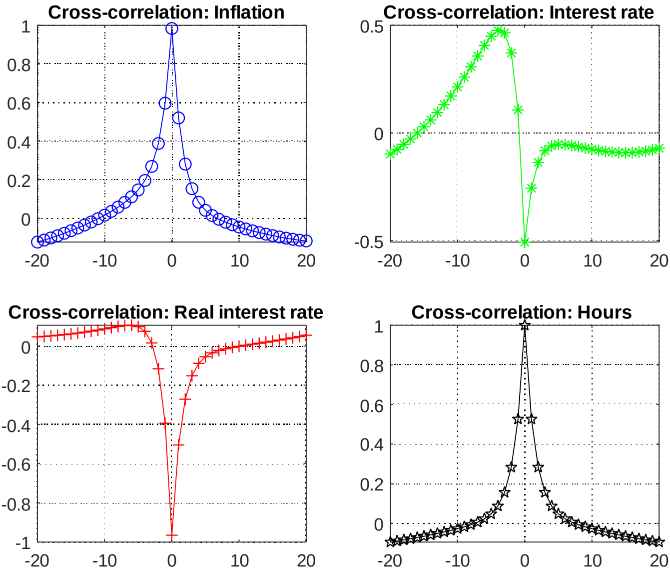
The NKM: IRF from both demand and shock shocks



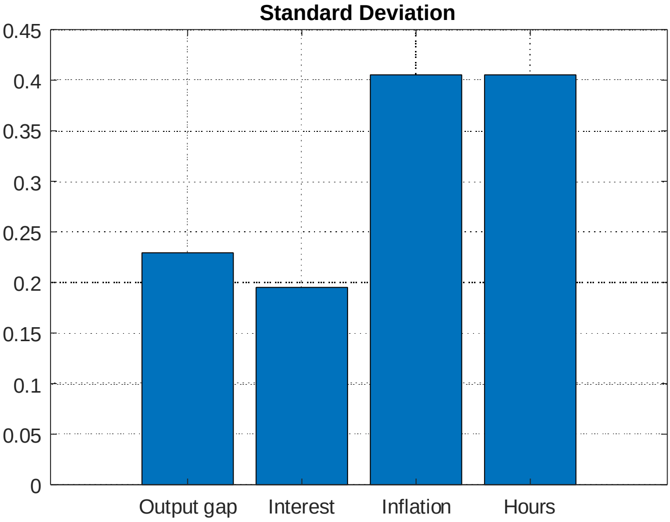
The NKM: IRF from both demand and shock shocks



The NKM: IRF from both demand and shock shocks



The NKM: IRF from both demand and shock shocks



IV – Recommended reading

Recommended reading

- Blanchard, O., and C.M. Kahn. (1980). The solution of linear difference models under rational expectations. *Econometrica* 48(5), 1305–1311. *This is the seminal paper on the the Blanchard-Khan approach*.
- Jordi Gali (2015). Monetary Policy, Inflation and the Cycle, 2nd Edition, Princeton University Press. *Look only at Chapter 3. This is the easiest treatment of the entire NKM with simple monetary policy rules.*
- Ellison, Martin (2009). Real Business Cycle Theory, mimeo, University of Oxford. *This is the best source I know, for showing in a very easy way what we really do when simulating a DSGE model, using the Blanchard-Khan approach.*