Introduction to Real Business Cycles: The Solow Model and Dynamic Optimization

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Summary

1. The Solow growth model revisited
2. The Solow growth model with dynamic optimization
3. Some examples of different types of dynamics in two dimensional systems
4. The dynamics of the Solow model (with optimization) step by step
5. Readings

What is a "Real Business Cycle" model?

The Real Business Cycle model is:

- The original **Solow growth model**, that we learn in a bachelor degree in economics

$$+$$

- inserted into a **dynamic optimization** framework: *no more a constant savings rate*

$$+$$

- With **shocks** to total factor productivity (the A variable)

$$+$$

- With **uncertainty** about those shocks

I — The Solow growth model revisited

The model in structural form

- The Solow model has six main equations

$Y_t = K_t^{\alpha}(A_t N_t)^{1-\alpha}$	,	production function
$K_t = K_{t-1} + I_{t-1} - \delta K_{t-1}$	,	capital accumulation
$I_t = sY_t$	,	investment
$N_t = (1+n)N_{t-1}$	,	labor accumulation (exogenous)
$A_t = (1+m)A_{t-1}$	,	technological progress (exogenous)
$Y_t \equiv C_t + I_t$	,	income accounting identity

- Parameters should satisfy: $0 < \alpha < 1, \delta > 0$, $0 < s < 1$, and n, m as exogenous growth rates that can be positive/negative

Too complex ... let's apply a trick

- **Structural form**: the model has **6** dynamic equations and one is nonlinear
- **Too complicated** to analyze the dynamics of a model like that
- **Apply a trick**: always try to *reduce the dimension* of the model
- **The trick pays off well:**, we can reduce the dynamics to just **one equation**
- **Incorporates the dynamics of all variables**, then we can use these results to obtain the dynamics of all other variables
- **Why?** Because labor and technology are exogenously accumulated in this model.

The Solow growth model in intensive form

- Rewrite the model by using variables defined in **intensive form**

$$y_t \equiv \frac{Y_t}{A_t N_t}, \quad k_t \equiv \frac{K_t}{A_t N_t}, \quad c_t \equiv \frac{C_t}{A_t N_t}, \quad i_t \equiv \frac{I_t}{A_t N_t}$$

- The dynamics of k_t encapsulates the dynamics of all variables: $K_t, A_t, N_t, Y_t, I_t, C_t$

- **The production function** in intensive form

$$y_t = k_t^{\alpha}$$

Proof.

$$y_t \equiv \frac{Y_t}{A_t N_t} = \frac{K_t^{\alpha}(A_t N_t)^{1-\alpha}}{A_t N_t} = \frac{K_t^{\alpha}(A_t N_t)^{-\alpha}(A_t N_t)^{1}}{(A_t N_t)^{1}} = \frac{K_t^{\alpha}}{(A_t N_t)^{\alpha}} = k_t^{\alpha}$$

□

The dynamics of capital accumulation

- We already know that

$$K_t = K_{t-1} + \underbrace{I_{t-1}}_{=s\cdot Y_{t-1}} - \delta \cdot K_{t-1} \quad (1)$$

- For simplicity, call the product of labor and technology $(A_t N_t)$ as **labor measured in efficiency units**, or simply E_t

$$E_t \equiv A_t N_t$$

- Divide both sides of eq. (1) by $E_{t-1} \equiv A_{t-1} N_{t-1}$

$$\frac{K_t}{E_{t-1}} = \frac{K_{t-1}}{E_{t-1}} + s\frac{Y_{t-1}}{E_{t-1}} - \delta\frac{K_{t-1}}{E_{t-1}}$$

- Now, **apply a trick** to the left hand side of the previous eq.

$$\frac{K_t}{E_t}\frac{E_t}{E_{t-1}} = \frac{K_{t-1}}{E_{t-1}} + s\frac{Y_{t-1}}{E_{t-1}} - \delta\frac{K_{t-1}}{E_{t-1}} \quad (2)$$

The dynamics of capital accumulation (continued)

- Apply the definitions of variables in intensive form to simplify our previous result

$$k_t \frac{E_t}{E_{t-1}} = k_{t-1} + s \cdot y_{t-1} - \delta \cdot k_{t-1} \tag{3}$$

- **Get rid of the term** $\frac{E_t}{E_{t-1}}$ above.
- If $E_t \equiv A_t N_t$, the gross growth rate of E_t is the product of the gross growth rates of A_t and N_t, that is

$$E_t = (1+m)(1+n)E_{t-1}, \quad \text{or}$$

$$\frac{E_t}{E_{t-1}} = (1+m)(1+n) \tag{4}$$

- Proof: If $E_t \equiv A_t N_t$ and $E_{t-1} \equiv A_{t-1} N_{t-1}$, then

$$\frac{E_t}{E_{t-1}} = \underbrace{\frac{A_t}{A_{t-1}}}_{(1+m)} \underbrace{\frac{N_t}{N_{t-1}}}_{(1+n)} = = (1+m)(1+n)$$

The dynamics of capital accumulation (cont.)

- Our system can be reduced to **one simple equation**
- Substitute the rate of growth of E_t given by equation (4) into equation (3), and get

$$k_t(1+m)(1+n) = k_{t-1} + s \cdot y_{t-1} - \delta \cdot k_{t-1} \tag{5}$$

- Now do three things:
  - subtract k_{t-1} from both sides
  - for simplicity define $(1+n)(1+m) = \phi$
  - take into account that $y_{t-1} = k_{t-1}^{\alpha}$ (see second slide of the Solow model)
- The result will appear as

$$k_t - k_{t-1} = \frac{1}{\phi}\left[(1-\delta-\phi)k_{t-1} + s \cdot k_{t-1}^{\alpha}\right]$$

The steady state of the Solow model

- The dynamics of the model can be represented by this single equation

$$k_t - k_{t-1} = \frac{1}{\phi}\left[(1-\delta-\phi)k_{t-1} + s \cdot k_{t-1}^{\alpha}\right]$$

- To solve for the **steady state** impose the condition $k_t = k_{t-1} = \bar{k}$.
- This implies that the left hand side of the previous eq. vanishes because $k_t - k_{t-1} = 0$ and the steady state can be obined from

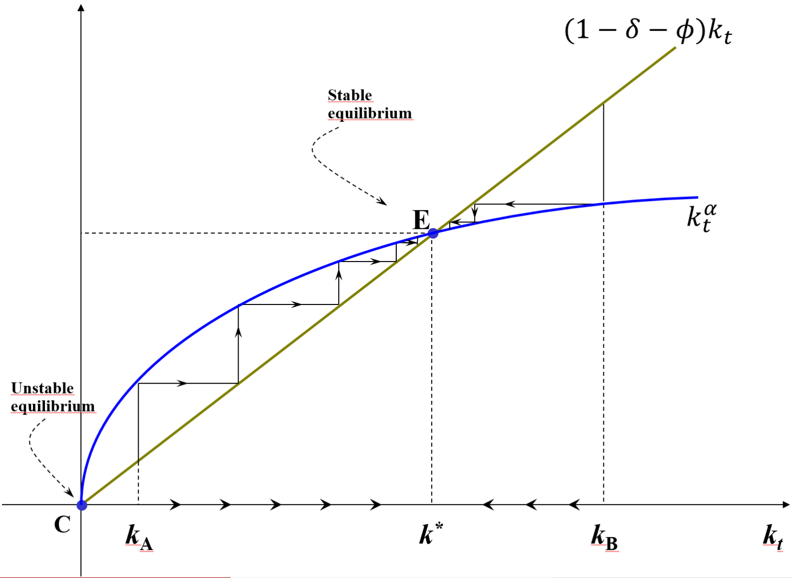
$$(1-\delta-\phi)\bar{k} = -s \cdot \bar{k}^{\alpha}$$

- Then, the value of the steady is equal to:

$$\bar{k} = \left[-\frac{s}{(1-\delta-\phi)}\right]^{\frac{1}{1-\alpha}}$$

- And now we get clear answers in economic dynamics: the LTE **exists, is stable, and unique**. (see next figure)

The steady state of the Solow model: graphical analysis



The steady state: growth rates

- What happens to the growth rates of the various variables in the model, **in the steady state**?
- We have three different classes of variables:
  - Variables measured in intensive form $(k, c, y,$ etc$)$
  - Variables measured in actual values $(K, C, Y,$ etc.$)$
  - Variables measured in per capita values $(K/N, C/N, Y/N,$ etc.$)$
- **By definition**, we know that in the steady state k remains constant over time $(k_t = k_{t-1} = \bar{k})$, so its growth rate has to be

$$g_k = 0$$

- What happens to the growth rate of capital (K)?
- What happens to the growth rate of capital per capita (K/N)?

The steady state: growth rates

- Calculate the growth rate of **Capital** (K). As $k = K/(AN)$, it is easy to show that

$$g_{(k)} \simeq g_{(K)} - \left(g_{(A)} + g_{(N)}\right)$$

- As we know that in the steady state $g_k = 0$, then

$$0 \simeq g_{(K)} - (m+n) \Rightarrow g_{(K)} \simeq m+n$$

- Let's turn now to the growth rate of **Capital per capita** (K/N) :

$$\begin{aligned} g_{(K/N)} &\simeq g_{(K)} - g_{(N)} \simeq (m+n) - n \\ &\simeq m \end{aligned}$$

- As an exercise, confirm that the following results hold generically:
    - **All variables** measured in intensive values grow at the rate: 0
    - **All variables** measured in per capita values grow at the rate: m
    - **All variables** measured in actual values grow at the rate: $m+n$

II — The Solow model with dynamic optimization

Introducing dynamic optimization

- **Two major limitations** are present in the the model just outlined, for the analysis of **business cycles:**
  - It assumes a constant savings rate
  - It does not incorporate any form of uncertainty
- **The first problem:** two major implications
  - **No consumprion smoothing/ no optimization**: private agents do not change their savings/consumption decisions in the face of changes in economic conditions, so no optimization is undertaken over time
  - **No welfare policy**, no results can be obtained (no utility analysis can be undertaken in this model)
- **Next: dynamic optimization** into the Solow model
- **We leave uncertainty for the next class**, when we get into the full Real Business Cycle model

The objective function

- **No more s constant**, no more $i_t \equiv s \cdot y_t$
- **Now s varies across time** in order to maximize the **welfare function** of private agents
- **The welfare function** (in this example) is only dependent on the utility obtained from **real consumption**
- **The utility should be discounted** to the initial period by applying a discount factor $\beta = 1/(1+r)$, where r is the discount rate

$$\beta^0 U(c_0) + \beta^1 U(c_1) + \beta^2 U(c_2) + \beta^3 U(c_3) + ...; \; t = 0, ..., \infty$$

- **The objective:** maximize the present discounted value of intertemporal utility

$$\max \sum_{t=0}^{\infty} \beta^t U(c_t)$$

- We have one major ingredient of an optimal dynamic problem: **the objective function.**

The constraints

- The economy has **three major constraints**
- ... the production capacity,

$$y_t = f\left(k_{t-1}\right)$$

- ... the capital accumulation

$$k_t = k_{t-1}\left(1-\delta\right) + i_t$$

- ... and the income accounting identity

$$c_t + i_t = y_t$$

- From those, we can derive **one single macroeconomic constraint** as in the Solow model without optimization

$$c_t + k_t = f\left(k_{t-1}\right) + \left(1-\delta\right)k_{t-1}$$

The optimal problem

- Two major ways to solve for the equilibrium of this process: a **decentralized equilibrium** and a **central planner equilibrium**
- Let's concentrate only on the social planner: maximizes the objective function subject to a resource constraint.
- In mathematical form this problem can be written as

$$\max_{\{c_t,k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$
$$s.t.$$
$$c_t + k_t = f(k_{t-1}) + (1-\delta) k_{t-1}$$

- This problem can be solved by **three alternative methods**:
  - Optimal control
  - Dynamic programming
  - Lagrangian multipliers
- We will use the Lagrangian method (more intuitive)

The Lagrangian

- For simplicity let's assume that **utility is logarithmic**: $U(c_t) = \ln c_t$, and the production function is **Cobb-Douglas**: $f(k_{t-1}) = k_{t-1}^{\alpha}$

$$\mathcal{L} = \max_{\{c_t, k_t\}_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty} \beta^t \{ \underbrace{\ln c_t}_{objective} - \lambda_t \underbrace{(c_t + k_t - k_{t-1}^{\alpha} - (1-\delta) k_{t-1})}_{constraint} \} \right]$$

- The **first order conditions** (FOC) should be taken with respect to the control variable (c_t), to the state variable (k_t) and to the Lagrangian multiplier (λ_t)
- Write down the Lagrangian function for two consecutive periods

$$\begin{aligned} \mathcal{L} \;&=\; ... + \beta^t \{ \ln c_t - \lambda_t (c_t + k_t - k_{t-1}^{\alpha} - (1-\delta) k_{t-1}) \} \\ &\quad + \beta^{t+1} \{ \ln c_{t+1} - \lambda_{t+1} (c_{t+1} + k_{t+1} - k_t^{\alpha} - (1-\delta) k_t) \} + ... \end{aligned}$$

First order conditions (FOC)

- Having

$$\begin{aligned}\mathcal{L} &= ... + \beta^t \{\ln c_t - \lambda_t (c_t + k_t - k_{t-1}^{\alpha} - (1-\delta) k_{t-1})\} \\ &+ \beta^{t+1} \{\ln c_{t+1} - \lambda_{t+1} (c_{t+1} + k_{t+1} - k_t^{\alpha} - (1-\delta) k_t)\} + ...\end{aligned}$$

- The FOCs are:

$$\begin{aligned}\partial \mathcal{L} / \partial c_t &= \beta^t \left(\frac{1}{c_t} - \lambda_t \right) = 0 \\ \partial \mathcal{L} / \partial k_t &= -\beta^t \cdot \lambda_t + \beta^{t+1} \cdot \lambda_{t+1} (\alpha k_t^{\alpha-1} + 1 - \delta) = 0 \\ \partial \mathcal{L} / \partial \lambda_t &= c_t + k_t - k_{t-1}^{\alpha} - (1-\delta) k_{t-1} = 0\end{aligned}$$

Simplifying the FOCs

- The dynamic system described by the three FOCs can be further reduced to just two equations of motion
- From the first FOC we know that as $\beta^t \neq 0$, then $\frac{1}{c_t} - \lambda_t = 0$, from where
$$\frac{1}{c_t} = \lambda_t, \qquad \frac{1}{c_{t+1}} = \lambda_{t+1}$$
- Now insert the values of λ_t and λ_{t+1} into the second FOC, and solve for c_{t+1} to get
$$c_{t+1} = \beta \cdot c_t(\underbrace{\alpha k_t^{\alpha-1}}_{=f'(k_t)} + 1 - \delta)$$
- So the dynamics of the model can be fully described by the previous eq. (known as the **EULER Equation**) and the third FOC, respectively
$$c_{t+1} = \beta \cdot c_t(\alpha k_t^{\alpha-1} + 1 - \delta) \tag{6}$$
$$c_t + k_t = k_{t-1}^{\alpha} + (1-\delta)\, k_{t-1} \tag{7}$$

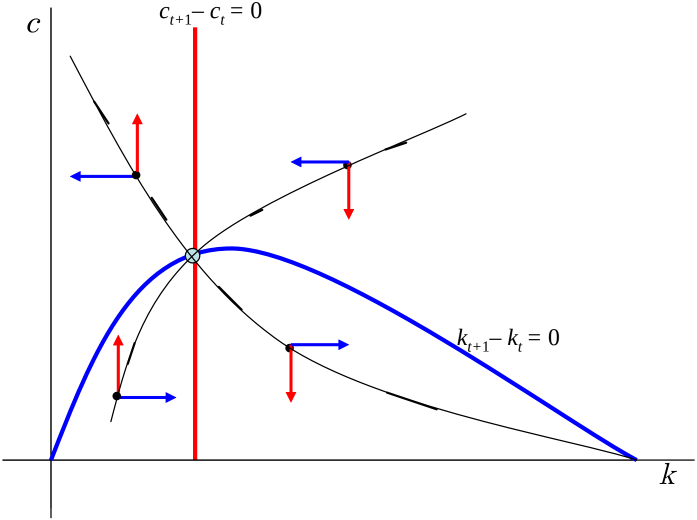
The steady state

- The steady state can be obtained by imposing the following two conditions to the system: eq. (6) and (7)

$$\begin{aligned} c_{t+1} &= c_t = \bar{c} \\ k_t &= k_{t-1} = \bar{k} \end{aligned}$$

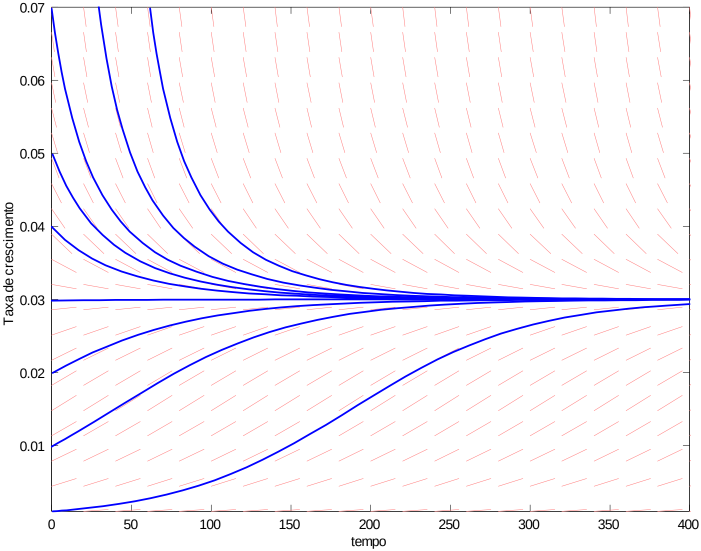
- It is easy to show that we have a unique steady state in this model. **Solve it as exercise. In order to simplify things assume that $\delta = \mathbf{1}$)**.
- However, the stability analysis is a little more complicated and shows that this steady state is unstable (or rather, is **saddle path stable**).
- See next figure for the dynamics of this model

An example of saddle path stability

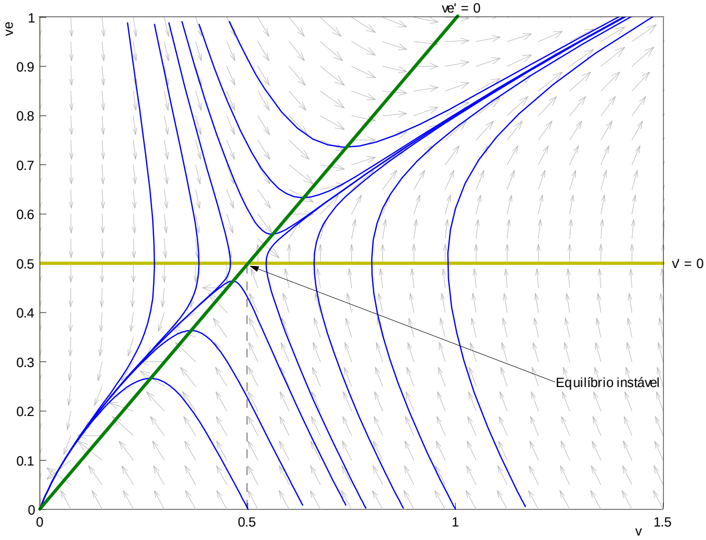


III – Some examples of different types of dynamics in two dimensional systems

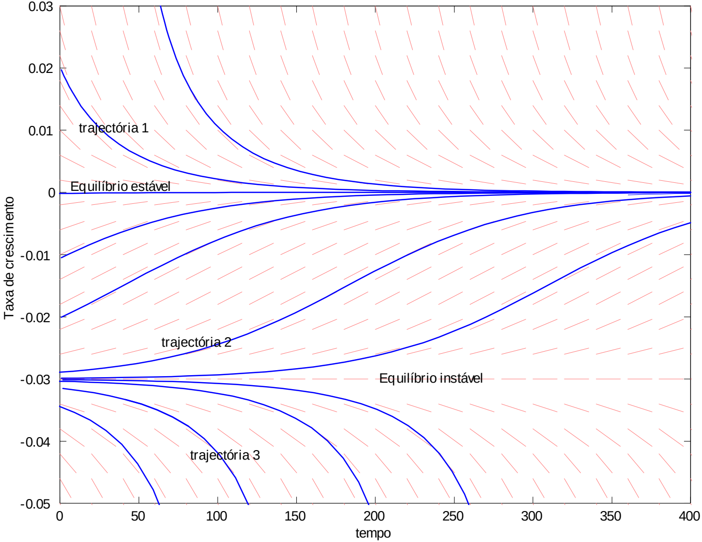
One stable equilibrium



One unstable equilibrium



Multiple equilibria: on stable, one unstable



IV — The dynamics of the Solow model (with optimization) step by step

How do we obtain such dynamics: Solow model

- The next six slides can be skipped without loss of relevant knowledge for the remaining part of the course[1]
- As we know, the dynamics of the model can be fully described by

$$
\begin{aligned}
c_{t+1} &= \beta \cdot c_t(\alpha k_t^{\alpha-1} + 1 - \delta) && (8) \\
c_t + k_t &= k_{t-1}^{\alpha} + (1-\delta)\, k_{t-1} && (9)
\end{aligned}
$$

- These eq. can be written in a more useful way. Subtract from both sides: c_t in the first eq. and k_{t-1} in the second eq., and rearrange

$$
\begin{aligned}
c_{t+1} - c_t &= \beta \cdot c_t(\alpha k_t^{\alpha-1} + 1 - \delta) - c_t \\
&= c_t \left[\beta(\alpha k_t^{\alpha-1} + 1 - \delta) - 1\right] && (10) \\
k_t - k_{t-1} &= k_{t-1}^{\alpha} - \delta k_{t-1} - c_t && (11)
\end{aligned}
$$

[1] They are useful for those who want to learn about stability in a system of two difference equations in a graphical framework (phase space)

The dynamics of c

- By imposing the condition $c_{t+1} = c_t = \bar{c}$ to eq. (10), we get an expression giving the steady state of variable c :

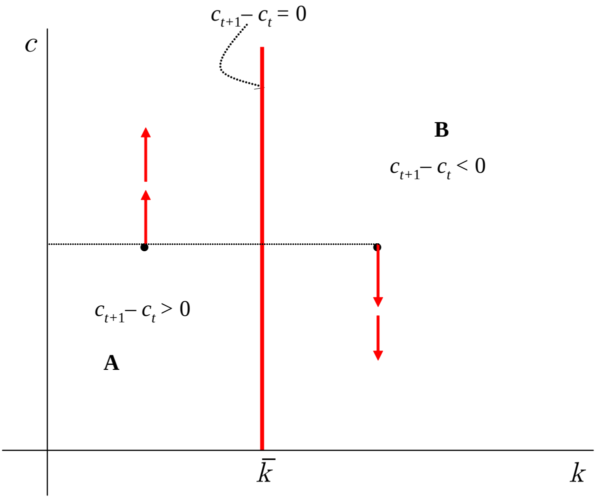
$$\bar{k} = \left[\frac{\alpha}{(1/\beta) - 1 + \delta} \right]^{\frac{1}{1-\alpha}}$$

- For this value of $\bar{k}$, c does not change over time, that is $c_{t+1} = c_t = \bar{c}$.
- Now notice that, if we choose some initial condition for k, that is k_0, then we have:
  - if $k_0 > \bar{k}$, then $c_{t+1} - c_t > 0$ and therefore c is growing over time $c_{t+1} - c_t$
  - if $k_0 < \bar{k}$, then $c_{t+1} - c_t < 0$ and therefore c is decreasing over time
  - if $k_0 = \bar{k}$, then $c_{t+1} - c_t = 0$ and therefore c remains constant over time
- Check this result trough a numerical example. Assume the following values for the parameters: $\beta = 0.95, \alpha = 0.4, \delta = 0.2$. For these values, $\bar{k} = 2.15$.

The dynamics of c (cont.)

- Now give three different values to k_0:
  - if $k_0 = 2.15$, then in eq. (10) $c_{t+1} - c_t = 0$ implies that $\beta(\alpha k_t^{\alpha-1} + 1 - \delta) - 1 = 0$. Check that for those parameter values this condition is satisfied.
  - if $k_0 = 4$, the term $\beta(\alpha k_t^{\alpha-1} + 1 - \delta) - 1 = -0.074 < 0$, which implies that $c_{t+1} - c_t < 0$, and therefore c is decreasing over time.
  - if $k_0 = 1$, the term $\beta(\alpha k_t^{\alpha-1} + 1 - \delta) - 1 = 0.14 > 0$, which implies that $c_{t+1} - c_t > 0$, and therefore c is increasing over time.
- Confirm the arrows directions in the next figure, which shows the dynamics of c over time for those values of k_0.

The dynamics of c (continued)



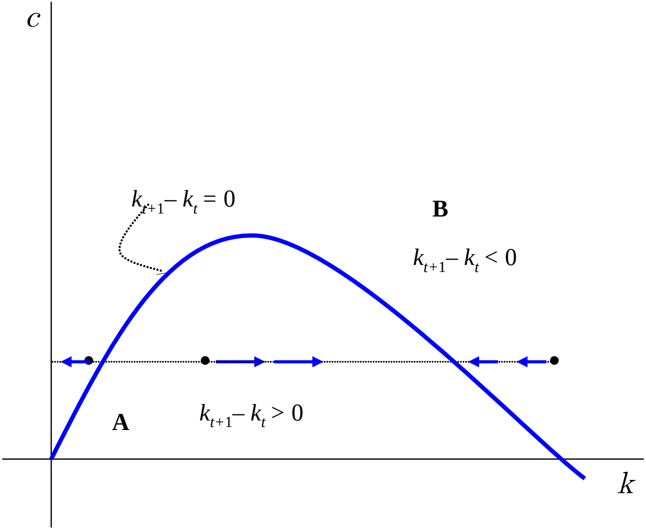
The dynamics of k

- By imposing the condition $k_t = k_{t-1} = \bar{k}$ to the eq. (11) we get an expression giving the steady state loci of variable k :

$$\bar{c} = \bar{k}^{\alpha} - \delta \bar{k}$$

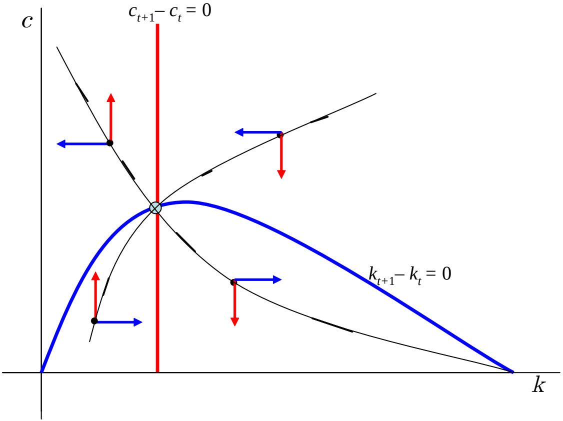
- In this case it is easy to see that:
    - if $\bar{c} > \bar{k}^{\alpha} - \delta \bar{k}$, then $k_t - k_{t-1} < 0$ and, therefore, k is decreasing over time.
    - if $\bar{c} < \bar{k}^{\alpha} - \delta \bar{k}$, then $k_t - k_{t-1} > 0$ and, therefore, k is increasing over time.
- The only difference is that the term $\bar{k}^{\alpha} - \delta \bar{k}$ is not linear, but this does not change in any way our basic points about the dynamics of the equation (11)
- See next figure for the steady state loci of variable k

The dynamics of k (continued)



Our example of saddle path stability

Putting together the two curves into one plane, we get two manifolds: one stable and one unstable; its the saddle path stability



Things to remember

1. How to obtain the fundamental equation of the Solow model
2. How to solve for the steady state using the fundamental equation of the Solow model
3. How to obtain the steady state graphically
4. In the steady state:
    1. All variables measured in intensive values grow at the rate: 0
    2. All variables measured in per capita values grow at the rate: m
    3. All variables measured in actual values grow at the rate: $m + n$

Things to remember (cont.)

1. What are the major limitations of the Solow model for the analysis of short term business cycles
2. What are the major ingredients of an optimal dynamic problem
3. The methods to solve this optimal dynamic problem are three: we use Lagrangean multipliers

V — Readings

Bibliography

- For the basic Solow growth model, any intermediate macroeconomics textbook will cover almost all basic issues.
- For the Solow model with dynamic optimization see:

Eric Sims (2017). Graduate Macro Theory II: Notes on Neoclassical Growth Model, University of Notre Dame.

Very good for the study of the Solow growth model: short and clear.

Dirk Krueger (2007). "Quantitative Macroeconomics: An Introduction" (Cap.3 and following), unpublished manuscript, Department of Economics University of Pennsylvania.

It's a long text. Very useful for studying the Solow growth model with dynamic optimization. However, given its length, it is better if this text is used as complementary material.